Hyperbolic Metamaterials at Microwaves with Stacked Inductive Coiled-Wire Arrays

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Abstract—Metasurfaces made of arrayed wires are inductive only for propagating waves and become capacitive when illuminated by evanescent waves. We show how to extend the inductive behavior of such metasurfaces to the case of illumination by plane waves with large wavenumber, i.e., for evanescent waves. The proposed metasurface exhibits an effective surface inductance that is extended to evanescent waves with large transverse wavenumber using coiled wires. Metasurfaces with inductive response over a wide spatial spectrum can be used for the realization of hyperbolic metamaterials (HMs) at microwave frequencies. Stacking such inductive metasurfaces separated by thin dielectric spacers leads to a negative real part of the, in-plane, effective dielectric permittivity, and thus to hyperbolic wavevector dispersion at microwave frequencies. We establish the requirements for such hyperbolic dispersion condition using arrays of coiled wires, also accounting for losses. We show that the negative in-plane real part of dielectric permittivity stretches up to transverse wavenumbers that are several times larger than the free-space wavenumber, over a broad frequency bandwidth. A theory and important design analytic formulas are presented to demonstrate the inductive behavior as well as the propagative nature of waves in the proposed HM, in agreement with full-wave simulations.

Index Terms—Hyperbolic metamaterial, absorbing media, anisotropic media, Metamaterials, frequency selective surfaces, super lens.

I. INTRODUCTION

HYPERBOLIC METAMATERIALS (HMs) are a class of artificial anisotropic media that allows for propagation of plane waves that would be otherwise evanescent in a homogenous dielectric medium. HMs are characterized by an anisotropic permittivity (or permeability) with real parts of opposite signs along two principal directions. Many applications benefit from a hyperbolic wavevector dispersion, such as enhancement of nearfield absorption from scatterers at the HM surface [1], suppression of reflection [2], increasing the decay rate of spontaneous emission [3], [4], negative refraction [5] and super-lensing [6], [7] or energy harvesting. In [8] the interplay of Floquet modes generated by an array on top of a HM and the absorbing channels of the HM has been studied. However, realistic designs of such metamaterials have only been proposed in the optical range, since nature offers materials with negative permittivity such as metals at optical frequencies [1]–[4], [6], [9], [10]. In terms of equivalent macroscopic properties of a homogenized medium, the effective relative permittivity tensor takes the form $\epsilon_\text{eff} = \epsilon_0 (\hat{x}\hat{x} + \hat{y}\hat{y}) + \epsilon_r \hat{z}\hat{z}$.

Here bold symbols define vectors and a hat defines unit vectors. A bar underneath denotes either a tensor or a matrix. Hyperbolic wave-vector dispersion is obtained when $\text{Re}(\epsilon_y) \text{Re}(\epsilon_z) < 0$. A metamaterial with such a macroscopic permittivity tensor, specifically $\text{Re}(\epsilon_y) < 0$ and $\text{Re}(\epsilon_z) > 0$, in a wide frequency band can be obtained by stacking inductive sheets and dielectric spacers in an alternating fashion as shown in Fig. 1: similar to what was proposed to realized graphene-dielectric based HMs operating in the far-infrared regime [11]–[14]. We also point out that 2D metamaterials, namely metasurfaces [15]–[18] have recently demonstrated new schemes toward enhancing light-matter interaction. Within such pervasive framework, the hyperbolic dispersion concept has been readily extended to flat metasurfaces [19]–[23].

![Fig. 1](a) A stack made of thin multilayered inductive metasurfaces described by surface conductivity $\sigma$, and dielectric spacers having thickness $h$ and relative dielectric constant $\epsilon_r$. Such a stack macroscopically behaves like a bulk anisotropic medium (right panel); and can be engineered in such a way to exhibit hyperbolic wavevector dispersion when $\text{Re}(\epsilon_y) \text{Re}(\epsilon_z) < 0$, for large plane wave transverse wavenumber $k_t$. (b) Ideal wavevector isofrequency dispersion relation $k_z = k_{t0}$ of waves in a homogenous HM that shows a hyperbolic shape for large transverse wavenumber $k_t > \sqrt{\epsilon_r} k_{t0}$. In this paper, we show the HM behavior for which $\text{Re}(\epsilon_y) > 0$ and $\text{Re}(\epsilon_z) < 0$, at microwave frequencies where the constituent metasurfaces are made of inductive coiled wire arrays. The arrows indicate the direction of the group velocity $v_g$ with respect to the wavevector, denoting backward propagating waves with large $k_t$.

At microwave frequencies, engineered metamaterials have been investigated [24], [25] mostly exhibit the desired properties for
the propagating part of the wavenumber spectrum \((k_i < k_0)\)
where \(k_i\) is the transverse wavenumber, \(k_0 = \omega/c\) is the
wavenumber in vacuum, \(\omega\) is the angular frequency and \(c\) the
speed of light in vacuum. Only very few architected designs at
microwave frequencies were proposed to engineer wave
properties for the part of the wavenumber spectrum that is
evanescent in vacuum (i.e., for \(k_i > k_0\)). For example, in [26] a
flat lens consisting of a uniaxial wire medium loaded with
metallic patches and lumped inductive loads is proposed.
However, the geometry of such lenses is not planar, and
therefore not practical for realizing multilayered metamaterials.

Enhanced absorption and high-spectrum propagation may
also be achieved using a double negative conjugate matched
layer as proposed in [27], yet this requires controlling both the
permittivity and permeability tensors.

In this paper, we propose for the first time a HM at
microwave frequencies made of stacked inductive metasurfaces
separated by dielectric spacers. The inductive metasurfaces are
composed of 1-D periodic arrays of coiled wires [see Fig. 1].
The proposed HM demonstrates a hyperbolic dispersion over a
wide range of wavevectors as seen in Fig. 1(b) as well as over a
wide frequency range as seen in Section V. Moreover, the
proposed metasurface has a low-profile geometry that could be
readily fabricated.

This paper is organized as follows. In Section II we state the
core principle: the requirement of layers to be inductive over a
wide spatial spectrum to synthesize a hyperbolic metamaterial.
Initially we resort to homogenous inductive sheets with
plane isotropy for simplicity. In Section III we investigate the
design principles to realize an inductive metasurface (also for
the evanescent spectrum) made of planar 1-D arrays of coiled
wires. We also validate the coiled wire’s required inductive
behavior via full wave simulations in Section IV and, using a
simplified model, we express the equivalent permittivity of the
stack. In Section V we demonstrate the hyperbolic behavior,
namely a negative real part of the transverse permittivity. The
hyperbolic behavior is achieved for \(f < 8\) GHz for \(k_i/k_0\) ratios
of several units.

II. CONDITION FOR HYPERBOLIC DISPERSION

We consider first an electrically thin metasurface in the \(x,y\)
plane, i.e., a single metasurface composing the stack of Fig.
1(a), and analyze its reactive behavior across the plane wave
spectrum. In this paper field quantities are in phasor domain
with time-harmonic convention \(e^{j\omega t}\) implicitly assumed and
suppressed in the following. We assume the metasurface is
illuminated by a plane wave \(e^{-jkr}\), where \(r\) denotes the
position vector, \(k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}\) is the wavevector, and
\(\sqrt{k_x^2 + k_y^2 + k_z^2} = k = \sqrt{\varepsilon_r} k_0\) is the
wavenumber in the host medium assumed to be isotropic and homogeneous with
relative dielectric constant \(\varepsilon_r\). The wavenumber in vacuum is
denoted by \(k_0 = \omega/c\). Here bold fonts denote vectors and hats
(\(^\wedge\)) denote unit vectors.

In general, a metasurface made of a planar array of metallic
wires with subwavelength period as in Fig. 2 is “homogenized”
in terms of its equivalent surface admittance, able to describe
its interaction with the plane-wave spectrum [28]. The
homogenized surface admittance of an array of metallic wires
typically exhibits inductive response only for the propagating
part of the plane wave spectrum, i.e., when \(k_i = \sqrt{k_x^2 + k_y^2} < k\)
and it becomes capacitive for large transverse wavenumbers
such that \(k_i = \sqrt{k_x^2 + k_y^2} > k\). In this paper, we propose an
effective approach to engineer metasurfaces wavevector
dispersion such that metasurfaces can preserve their inductive
behavior for a large range of transverse wavenumber in the
plane-wave spectrum with \(k_i = \sqrt{k_x^2 + k_y^2} > k\), upon loading
each wires with a proper distributed inductance as elaborated in
details in Section III.

Let us assume a metasurface located at \(z = 0\) (with negligible
thickness) and described by a homogenized 2-D tensorial (i.e.,
anisotropic) surface admittance \(\sigma_s\) (comprising \(\hat{\sigma}_{xx}\), \(\hat{\sigma}_{yy}\),
and \(\hat{\sigma}_{yx}\) entries) defined as
\[
\hat{\sigma} \times \left[ H_1(0^+) - H_1(0^-) \right] = \sigma_s \cdot E_s(0) \tag{1}
\]
where \(z \to 0^+\) and \(z \to 0^-\) denote locations infinitesimally
above and below the metasurface, respectively (the \(x\) and \(y\)
position dependences are implicit) and \(H_1\) and \(E_s\) are the total
transverse magnetic and electric fields. The subscript \(t\) denotes
transverse components (i.e., \(x\) and \(y\) components) of the
macroscopic averaged fields. When a metasurface is made of
orthogonally oriented arrays with a square lattice grid, it can be
described with a good approximation (except for very large \(k_i\)
spectrum) by a scalar admittance, i.e., \(\sigma_s = \sigma_s' \hat{\sigma} + j \sigma_s''\hat{\sigma} \)
with \(\sigma_s' = \sigma_s'' + j \sigma_s''\) that assumes complex values, and it is thus
in-plane isotropic. Negative values of \(\sigma_s''\) denote inductive
behavior and the positive values of \(\sigma_s''\) represent losses in the
metasurface.

We then consider a stack made of alternating metasurfaces
made of arrays of loaded wires separated by dielectric layers
with relative permittivity \(\varepsilon_d = \varepsilon_d' - j \varepsilon_d''\) and thickness \(h\) as
depicted in Fig. 1. Assuming first, for simplicity, a stack as in Fig. 1 where metasurfaces are modeled by an isotropic homogenized surface admittance \( \sigma = \sigma' + j\sigma'' \) with \( \sigma' \geq 0 \), we describe the equivalent relative permittivity tensor \( \varepsilon_r = \varepsilon' (\hat{x} \hat{x} + \hat{y} \hat{y}) + \varepsilon_z \hat{z} \hat{z} \) of the metamaterial by using the simple homogenization scheme valid for subwavelength [11], [12], [24]. The relative permittivity tensor entries are given by

\[
e_{i} = e'_{i} - j\sigma'_{i} = e_{d} - j\frac{\sigma'_{z}}{\omega \varepsilon_{0} h}, \quad e_{z} = e_{d}
\]  

where \( \varepsilon_{0} \) is the vacuum permittivity and \( \omega \) is the angular frequency. The \( \varepsilon_{zz} \)-entry of the effective permittivity is straightforwardly found as \( e'_{z} = e_{d} > 0 \) [11], since the metasurface host electric current has only transverse-to-\( z \)-components. Plane waves inside the metamaterial are decomposed into transverse-magnetic (TM\( ^{\sigma} \)) and transverse-electric (TE\( ^{\sigma} \)) waves, with respect to \( z \); whose wavevector dispersion is given by

\[
\frac{k_{x}^{2} + k_{y}^{2}}{\varepsilon_{t}} = k_{0}^{2} \quad \text{(TE\( ^{\sigma} \), ordinary waves)}
\]

and

\[
\frac{k_{x}^{2} + k_{y}^{2}}{\varepsilon_{z}} = k_{0}^{2} \quad \text{(TM\( ^{\sigma} \), extraordinary waves)}
\]

respectively. TM\( ^{\sigma} \) waves possess a propagating spatial spectrum

\[
k_{z} = \sqrt{\varepsilon'_{z} k_{0}^{2} - \frac{\varepsilon_{z} - k_{0}^{2}}{\varepsilon_{z}}} \quad \text{(TM\( ^{\sigma} \), extraordinary waves)}
\]

for any \( k_{t} > \sqrt{\varepsilon'_{z} k_{0}} \) when the real part of the transverse permittivity is negative, i.e., when \( e'_{z} < 0 \). This is illustrated in Fig. 1(b) where we show a typical isofrequency dispersion diagram pertaining to this case, when \( e'_{r} < 0 \) and \( e'_{z} > 0 \). In particular, the spectrum is propagating for \( k_{t} > k = \sqrt{\varepsilon'_{z} k_{0}} \), that would be otherwise evanescent in a regular isotropic material. Instead, TE\( ^{\sigma} \) waves are mainly evanescent when \( e'_{z} < 0 \) in the entire spatial \( k_{t} \) spectrum. Table I summarizes the evanescent/propagating spectral ranges and the conditions of the dispersive permittivity tensor entries.

| TABLE I. THE CONDITION OF TM\( ^{\sigma} \) PROPAGATION IN IDEAL LOSSLESS UNIAXIAL ANISOTROPIC MEDIUM |
|---|---|
| \( k_{t} < k \) | \( k_{t} > k \) |
| \( e'_{z} < 0 \), \( e'_{z} > 0 \) | Propagating |
| \( e'_{z} > 0 \), \( e'_{z} > 0 \) | (Desired Case) |
| \( e'_{z} < 0 \), \( e'_{z} > 0 \) | Evanescent |

According to (2), a negative value of the real effective permittivity \( e'_{z} = e'_{z} + \sigma''/(\omega \varepsilon_{0} h) \) for the homogenized medium in Fig. 1, is obtained when

\[
\frac{\sigma''}{\omega \varepsilon_{0} h} < -\varepsilon'_{z}.
\]

The imaginary part of the relative permittivity representing losses in the effective medium is given by \( e''_{z} = e''_{d} + \sigma''/(\omega \varepsilon_{0} h) \). To keep simple analytic formulae, we will firstly assume that the dielectric spacer separating the metasurfaces has negligible dielectric losses \( e''_{d} = 0 \) and we consider that losses are only due to the ohmic losses in the loaded wires as discussed in Section III. Therefore losses in the homogenized bulk material are described mainly by \( e''_{z} = \sigma''/(\omega \varepsilon_{0} h) \). In summary, if one desires to have a material with \( e'_{z} < 0 \), each metasurface should be sufficiently inductive as described by the inequality to be satisfied by \( \sigma'' \) in (6).

Condition (6) would ensure a hyperbolic \( k_{t} - k_{z} \) isofrequency dispersion diagram shown in Fig. 1(b), supporting propagative waves for \( k_{t} > k \), if losses are not undesirably high as discussed next. Strikingly, a stack as in Fig. 1 would not provide a hyperbolic dispersion diagram if the wires in the array are just made of straight perfect electric conducting (PEC) wire, because due to spatial dispersion the PEC wire arrays lose their inductive behavior for \( k_{t} > k \) (see page 284 in [29]). Here we show a realistic scheme with intrinsically reactive wires and we also take into account ohmic losses, which could detrimentally affect the hyperbolic propagation spectrum in the stack in Fig. 1.

### III. INDUCTIVE METASURFACE FOR LARGE WAVENUMBERS

In this section we describe the inductive response of a metasurface of coiled wires for large plane wave spatial spectrum. From here on out, we refer to an infinite 1D linear conductive wire with a distributed line impedance simply as a “loaded wire” (See Fig. 2). The realistic implementation of an inductive metasurface uses “coiled wires” that refer to infinite wound metallic wires (See Fig. 3). Each coiled wire is then modeled as a uniform “equivalent loaded wire”, thus a 1-D line conductor with a uniformly distributed line impedance. We call “wire array” a metasurface made of wires along the \( x \)-direction, periodically spaced along the \( y \)-direction as in Fig. 2.

#### A. UniAxial Wire metasurface: Need of an Inductive Load

The metasurface made of 1-D periodically arrayed wires oriented along \( x \), located at \( z = 0 \) as in Fig. 2, is illuminated by a plane wave with incident electric field vector \( \mathbf{E}^{\text{inc}} e^{-jk_{t} \mathbf{r}} \). Each loaded wire is described by its impedance per unit length \( Z_{t} \) and it is assumed to have negligible diameter compared to the wavelength and the distance from other wires. Therefore, neglecting the wires’ transverse size, the current flows along \( x \), hence only the \( x \) component of the electric field interacts with the wire array. For simplicity in the following we consider TM\( ^{\sigma} \) plane waves with \( x \)-directed electric fields and wavevector in the \( x \)-\( z \) plane (i.e., \( k_{t} = 0 \)). Subsequently, we assume a homogenization scheme in which the averaged transverse electric and magnetic fields are related at \( z = 0 \) through the boundary condition
where $\sigma_s$ is the effective homogenized surface admittance of the metasurface. This equation is equivalent to assuming that the surface hosts a homogenized surface current density $J_{s,x} = H_y(0^+) - H_y(0^-)$ with $J_{s,x} = \sigma_s E_x(0)$. The objective of the subsequent analysis is to obtain an expression for $\sigma_s$ in the case of “loaded wires array”.

The current along each wires is conveniently described by $Z_l I_x = E_x^{\text{loc}}$, where $I_x$ is the wire current projected along the $x$ direction and $E_x^{\text{loc}}$ is the local field acting on the same wire [28]. The wires are separated by a distance $d$, where we assume that the wires radius $\rho_0 << d$.

In [28] a closed form expression for the homogenized surface impedance $Z_{s,x} = 1/\sigma_s$ (called $Z_s$ in [28], page 81) is provided as a function of wavevector $k$ of the incoming plane wave that reads,

$$Z_{s,x} = Z_l d + j \frac{\eta}{2} \alpha \left(1 - \frac{k_z^2}{k^2}\right)$$

(8)

where $\eta = \sqrt{\mu / \varepsilon}$ the wave impedance in the host medium, and $\alpha$ is defined as

$$\alpha = \alpha_{ABC} + \frac{kd}{2\pi} \Lambda$$

(9)

with

$$\Lambda = \sum_{n=-\infty}^{\infty} \left[ \frac{2\pi}{\sqrt{2\pi n + k_z d}^2 - (k_z^2 - k^2) d^2} - \frac{1}{n} \right]$$

(10)

and $\alpha_{ABC} = (kd / \pi) \ln \left( d \left/ (2\pi \rho_0) \right. \right)$. Here we extend the expression (8) to the case of evanescent spectrum that is of main importance in our contest.

According to [28], [30], the infinite summation term $\Lambda$ in (10) can be neglected in (9) for a non-sparse array (i.e., made of dense wires with subwavelength period), hence $\alpha \approx \alpha_{ABC}$ is a good approximation. In fact, one can also neglect $\Lambda$ when considering the evanescent spectrum with $k_z > k$ as long as

$$k_z d << 2\pi$$

(11)

In particular, for the example wire array with period $d = 1$ mm, and wires’ radius $\rho_0 = 120$ µm the relative effect of the second term on the right hand side of (9) is negligible for frequencies of several GHz and transverse wavenumbers of several times the wavenumber in the host medium. Quantitatively, taking into account or neglecting $\Lambda$ leads to a difference in the real part of the surface conductivity $\sigma = 1/Z_{s,x}$ of 10% at the extreme [frequency/transverse wavenumber] range of the desired metasurface inductive behavior for evanescent fields, namely for $f = 10$ GHz and $k_z = 10 k$ (see Section V, Fig. 13). At lower frequencies or lower transverse wavenumber, the $\Lambda$ term is even more negligible. The frequency range at which the metasurface is inductive for an evanescent spectrum will determine the frequency range of operation of the hyperbolic material as it will be shown in Section IV and V.

The dependence of $\Lambda$ on $k_x$ and $k_y$ in (10) implies spatial dispersion but in dense array satisfying (11) its effect is less pronounced than the spatial dispersion given by $\left(1 - k_z^2 / k^2\right)$ in (8), hence it can be neglected. Following these assumptions (8) is rewritten by only taking $\alpha \approx \alpha_{ABC}$ as

$$Z_{s,x} \approx Z_{PEC}^0 \left(1 - \frac{k_z^2}{k^2}\right) + Z_l$$

(12)

where $Z_{PEC}^0 = 1/2 j \eta \alpha_{ABC}$ is the reactive sheet impedance of a metasurface made of PEC wires (i.e. when the distributed impedance vanishes, $Z_l = 0$) for $k_z = 0$, i.e., for normal incidence.

It is important to stress that in the case of PEC straight wires, where we do not have loading per unit length, i.e., $\text{Im}(Z_l) = 0$, the metasurface is inductive, i.e., $\text{Im}(Z_{s,x}) > 0$, only in the spectral region such that $k_z^2 < k^2$. Indeed, from (8) it is clear that the term $\left(1 - k_z^2 / k^2\right)$, which represents spatial dispersion of the surface impedance, leads to a changes of sign of $\text{Im}(Z_{s,x})$ for $k_z^2 > k^2$. This is the reason why the metasurface becomes capacitive for $k_z^2 > k^2$ if the wires are not inductively loaded, i.e., if $Z_l = 0$. As explained in [14], this capacitive effect introduced by the term $-\text{Im}(Z_{PEC}^0 (k_z^2 / k^2))$ is due to an accumulation of charges along the wires. Therefore metasurfaces made of PEC arrays cannot extend inductive behavior to large spatial spectrum and a stack of such metasurfaces fails to support hyperbolic dispersion. To reduce the relative weight of this capacitive effect, Ref. [25] proposes either to add periodic conductive plates normal to the wires (to spread the charges) or to enhance the natural inductive behavior of the wires (by locally coating the wires with a high permeability medium).

Here we are interested in having a metasurface with inductive behavior for the evanescent spectrum, i.e., we want to realize $\text{Im}(Z_{s,x}) > 0$ for $k_z^2 > k^2$. In this regard, the inductivity of the wires in $\text{Im}(Z_l)$ in (9) pushes further the spectral limit inductive response of wire arrays towards large transverse wavenumber such that $k_z^2 / k^2 < 1 + \text{Im}(Z_l) / \text{Im}(Z_{PEC}^0)$. An inductive $Z_l$ is realized by having a wire made of inductive coils discussed in the following. We also note that a dense array of thin wires, i.e. small $Z_{PEC}^0$, contributes to our goal and softens the need of inductive loading $Z_l$.

Next, we propose a realistic design using an inherently
inductive wire, including realistic losses due to the resistivity of the metal (copper). Then, we will show that this leads to metasurfaces that are inductive for the evanescent spectrum, which is a building block to engineer a HM, as shown in Fig. 3.

B. Inductive loading of a single wire

The proposed procedure to engineer the metasurface inductive impedance in (12) for the evanescent spectrum is based on having wires with distributed impedance $Z_l$ with a dominant inductance per unit length. In this paper, to realize such an inductive load, we focus on a realization based on winding a copper wire of radius $r_w$ as shown in Fig. 3. The overall external radius of the coil is $r_0$ with a pitch (period along the $x$-direction) denoted by $p$ and the corresponding number of pitches (periodic) per unit length is defined by $N_l = 1/p$. We will show that the coiled wire behaves as an “equivalent loaded wire” of radius $r_0$ with a proper distributed impedance $Z_l$. The design of inductance and resistance and their effect on the overall inductive behavior of the metasurface are discussed in this sub section. The frequency range chosen to illustrate the numerical example is [0.1 – 10] GHz. The dimensions of the coil and the stack spacing are chosen as annotated in Fig. 4.

![Coiled wire](image)

Fig. 3. The inductive wire is the building block for a metasurface that exhibits inductive behavior for the evanescent spectrum as discussed in Sec. III. In turn, such metasurface is a building block to engineer a HM made of stacking layers of coiled wire arrays, with dielectric spacers as discussed in Sec. IV.

![Coiled wire parameters](image)

Fig. 4. Coiled wire geometry and parameters of the coiled wire utilized in the numerical examples in this paper.

Let us now derive $Z_l$, namely the equivalent per-length impedance of such a coiled wire directed along $x$, using its equivalent circuit model. We assume here a current $I_w$ in the wound conductor of the coil flowing in a helical path (Fig. 4). The coiled wire is assumed to be infinite therefore we use an expression for the (series) inductance per unit length of the coil given by $L_x = \mu N_l^2 \pi (r_0 - 2r_w)^2$. Here $\mu$ is the permeability of the host medium. In addition, the coil has a (series) per unit length resistance $R_c$ and (shunt) stray capacitance $C_{sc}$. The potential difference along an arbitrary distance $\Delta x$ along the coiled wire (see Fig. 4) is simply obtained by applying Ohms law to the equivalent per unit length impedance of the coiled wire namely $Z_c = (R_c + j\omega L_x) || (j\omega C_{sc})^{-1}$, leading to

$$U = I_w \Delta x \frac{R_c + j\omega L_x}{1 + jR_c C_{sc} \omega - L_x C_{sc} \omega^2}$$

Initially we will assume the stray capacitance $C_{sc}$ is negligible. We investigate the effect of stray capacitance in the Section III.C.

The expression of $L_x$ is not considered as a function of $k_x$ (for an $x$-directed coil) as long as $k_x r_0 \ll 2\pi$ or $k_x / k \ll \lambda / r_0$. In the examples provided here, these conditions apply in the spectrum $k_x / k \ll 200$ at 10 GHz (this condition is fulfilled in the transverse wavenumber range of interest in Section V).

We are interested in the per-unit-length impedance that relates the local electric field and the local current, both of which are along the coiled wire axis (here the $x$ axis). The electric field along the $x$-axis is simply given by $E_{loc} = -\partial U / \partial x = -\Delta U / \Delta x$. The current flowing in the $x$-direction, $I_x$ is calculated by projecting the helical current $I_w$ onto the $x$ axis (wire axis) as

$$I_x = I_w \cos(\beta_{coil})$$

where $\beta_{coil} = \tan^{-1}(2\pi N_l (r_0 - r_w))$. Based on (13), and neglecting the stray capacitance, one has

$$E_{loc} = Z_{Iw} I_w = (R_c + j\omega L_x) I_w = Z_{Ix} I_x$$

Using (14) into (15) the resistance and the inductance per unit length are found as

$$Z_l = Z_{sc} \sec(\beta_{coil}) = R_l + j\omega L_l = (R_c + j\omega L_x) \left( 1 + 4\pi^2 N_l^2 \frac{r_0 - r_w}{(r_0 - 2 r_w)^2} \right)^{1/2}.$$  \hspace{1cm} (16)

We first evaluate the inductance $L_d$ by substituting the expression for $L_x$ into (16) leading to $	ext{Im}(Z_l) = j\omega L_l$, with the inductance per unit length given by

$$L_d = L_x \sec(\beta_{coil}) = \mu_0 N_l^2 \pi (r_0 - 2r_w)^2 \left( 1 + 4\pi^2 N_l^2 \frac{r_0 - r_w}{(r_0 - 2 r_w)^2} \right)^{1/2}$$

(17)

Considering the numerical values in Fig. 4, the projection term $\left( 1 + 4\pi^2 N_l^2 \frac{r_0 - r_w}{(r_0 - 2 r_w)^2} \right)^{1/2}$ is approximately $7$, and $L_d \approx 36.4 \mu$H/m.
C. Effect of loss and stray capacitance.

Ohmic losses due to the conductivity of copper can be accurately estimated using the skin depth, circumference and the curvilinear length of the wire winding in the frequency range of interest (the skin depth is maximum and equal to 4.6 μm at the minimum frequency of operation used in section VI, i.e., 0.2 GHz). Considering the resistance of copper wire of length equal to the curvilinear length of the helicoid, the resistance per unit length of coil is

$$ R_c \approx \frac{\mu_0}{2\pi} \rho_{Cu} \left( 1 + 4\pi^2 N^2 \right) \left( r_0 - r_w \right)^2 \right)^{1/2} $$(18)

where $\rho_{Cu}$ (Ω·m) is the resistivity of copper. This leads to the following ohmic resistance per unit length $R_i = \text{Re}(Z_i) = R_c \sec(\beta_{coil})$ of the equivalent straight wire that reads

$$ R_i = \frac{\mu_0}{2\pi} \rho_{Cu} \left( 1 + 4\pi^2 N^2 \right) \left( r_0 - r_w \right)^2 $$

(19)

With the numerical values chosen in Fig. 4 we obtain $R_i \approx 4521 \sqrt{f_{GHz}}$ Ω/m, where $f_{GHz}$ is the frequency in GHz.

In order to assess the effect of the stray per-unit-length capacitance $C_{sc}$, also known as the parasitic capacitance, we resort to the estimation of the parasitic capacitance of a solenoidal coil given for instance in [31]. For a "dense" coiled wire (such that $p \leq 4r_w$), the only contribution to $C_{sc}$ is the turn-to-turn (first neighboring turn) capacitance term denoted by $C_{tt}$. Therefore, the stray capacitance can be readily obtained via [31, Eqs. (1) and (12)]

$$ C_{sc} \approx C_{tt} = \frac{2\pi^2 \varepsilon_0}{N_j} \frac{p}{\ln \left( \frac{p}{2r_w} + \frac{p}{2r_w} \right)} $$

(20)

The condition $p \leq 4r_w$ is satisfied in our numerical example. For $p > 4r_w$, $C_{sc}$ based on (20) is overestimated by considering $p = 4r_w$ and as a conservative approach we can use this overestimated $C_{sc}$ in order to assess a margin for the stray capacitance to be negligible, i.e., without any significant impact on wave propagation characteristics. With the numerical values chosen (see Fig. 4), the formula in (20) leads to $C_{sc} \approx 1.44 \times 10^{-18}$ F·m. The self-resonance frequency of the coil due to the stray capacitance and the inductance per unit length is given by $f_{SRF} = \left( 2\pi \sqrt{L_i C_{sc}} \right)^{-1} \approx 22$ GHz that is higher than the operational frequencies in section V. This means that the stray capacitance has a negligible effect towards the inductive properties of the coil in the frequency range of hyperbolic dispersion obtained in the Sections IV and V.

In particular, at $f = 10$ GHz, considering or neglecting the stray capacitance $C_{sc}$ in (13) leads to a difference of less than 2.5% (resp. 0.02%) on the real part (resp. imaginary part) of the per-length impedance $Z_l$. This parasitic contribution is even smaller at lower frequencies, thus the stray capacitance has negligible effects along the entire frequency range considered in Sections IV through V.

In summary, the per-unit-length impedance of the equivalently loaded wire made of copper with the parameters in Fig.4 can be estimated by

$$ Z_l \approx \left( 4521 \sqrt{f_{GHz}} + j \left( 229 \times 10^3 f_{GHz} \right) \right) \Omega/m. $$

(21)

D. Full-wave validation of the coil model for a loaded wire

In this section we validate the inductive behavior of the coiled wire by performing full-wave simulations and extracting its equivalent inductance. For simplicity, we assume here a finite-length PEC coiled wire (in full-wave simulation, we assume a coil length that is much larger than the pitch, typically $>30p$). Fig. 5 shows the geometry of the coil for which the equivalent load distributed impedance needs to be extracted. As discussed above, we recall the definitions of equivalent impedance per unit length of the inductor with respect to both the equivalent wire current $I_s$ and the current $I_w$ flowing in the helical coiled wire in (15). We summarize here the steps carried out in order to retrieve the impedances $Z_w$ and $Z_l$ in (15) by evaluating the local fields and currents from full-wave simulations. We perform full-wave simulations of the coiled wire in Fig.5, assuming an incident plane wave polarized along the $x$-direction and impinging normally on the coiled wire (the plane wave wavenumber is incident long the $z$-direction). Here, and only in this subsection, we use the frequency-domain finite-element method implemented in HFSS version 15 by Ansys, Inc. Note that using an obliquely incident plane wave will not alter the result since the coil winding pitch $p$ is extremely subwavelength. The coiled wire length here is taken to be 35. To calculate the impedances $Z_w$ and $Z_l$ in (15), we first calculate the resulting helical current excited on the surface of the coiled wire due to the plane wave excitation, which is retrieved from full-wave simulations as

$$ I_w = \int_{C_1} \mathbf{H} \cdot d\mathbf{l} $$

(22)

where $\mathbf{H}$ is the total magnetic field resulting from full wave simulations and $C_1$ is a contour going around the coil conductor, as seen in Fig. 5. Moreover, the current $I_s$ in (14) is calculated from full-wave simulations using Ampere’s law including the displacement current as

$$ I_s = \int_{C_2} \mathbf{H} \cdot d\mathbf{l} - j\omega \varepsilon_0 \int_S \mathbf{E} \cdot d\mathbf{s} $$

(23)
where $E$ is the total electric field, $C_2$ is a circular contour in the transverse plane to the coil axis enclosing it with a radius $r_0$, and $S$ the surface area bounded by $C_2$.

![Fig. 5. Geometry of the coiled wire and inductance extraction scheme using full wave simulation. The equivalent current along the wire axis is $I_x$ whereas the current $I_w$ flows along the helical coil. Note that the counter $C_2$ is taken as a circular path in the azimuthal ($\phi$) direction.](image)

In Fig. 6 we plot the extracted equivalent impedance of the coil versus frequency for the dimensions in Fig. 4, calculated using the finite element method solver in frequency domain implemented in HFSS by Ansys, Inc. For convenience we show the imaginary parts of both $Z_l$ and $Z_w$. We also superimpose the result of the analytical formula in (17) where we observe good agreement between full-wave simulation and the analytical formulas for both $Z_l$ and $Z_w$, in the frequency range from 0.5 GHz to 15 GHz. The maximum error between the full-wave and the analytical formulas occurs at the highest frequency, i.e., at 15 GHz, there is ~5% error in the calculated inductance from full-wave simulation. Furthermore, the analytical model predicts slightly lower inductivity than the full-wave simulated values, hence calculations based on (17) would provide a conservative value for $\text{Im}(Z_l)$. The impedance exhibits a very linear behavior with frequency, even at 15 GHz, thus the stray capacitance estimated in (18) can be conveniently ignored in this frequency range as also expected from the theoretical model.

![Fig. 6. Imaginary part of per unit length impedance of the coiled wire $\text{Im}(Z_l)$ calculated from the analytic formula (16) and compared to the extracted reactance from full-wave simulation varying as a function of frequency.](image)

The analytical model for the inductance per unit length to be used in the metasurface equivalent impedance assumes that a coil can be modelled as a cylinder loaded with a distributed inductance. Therefore, to check that the electromagnetic field distribution around such a loaded cylinder is similar to that around a coiled wire, we use an equivalent lumped inductance model in full-wave simulations as shown in Fig. 7. A surface impedance boundary condition is assigned only to the outer surface of a cylinder of radius $r_w$ in the full-wave simulations (using HFSS by Ansys, Inc. “lumped RLC boundary” condition option in the finite element simulation) as seen in Fig. 7(b). The “lumped RLC boundary” condition in HFSS is related to the surface impedance boundary condition, as discussed in [32], and can be used to represent any combination of lumped/distributed resistance, inductance, and/or capacitance in parallel on a surface [33]. Such lumped RLC boundary condition enforces the relationship between the total tangential fields on the outer surface of the cylinder (excited with an external source) such that the resulting fields outside the cylinder would resemble those emanating from the coiled wire. The value of the lumped RLC boundary inductance used in the simulation in Fig. 7(b) is assumed to be equal to the per unit length inductance of the infinite coiled wire from (17) multiplied by the length of a finite-length cylinder (taken here as $35\rho$, equal to the length of the coiled wire) considered in the simulation to emulate the distributed nature of the inductance of the coiled wire. (An equivalent way to model the distributed inductance in HFSS is by using the surface impedance boundary condition in units of Ohm/$\mu$m, by specifying the value of the surface impedance at each frequency, which is calculated as the angular frequency multiplied by the inductance per unit length of the coiled wire multiplied by the length and divided by the perimeter of the cylinder). Excitation with a normally incident plane wave with electric field polarized along $x$ is used in both coiled wire and cylinder with inductive boundary for comparison purposes. We show the scattered electric fields at 5 GHz for the coil and the equivalent lumped model. A good agreement of scattered field profile outside the wire and coil between both cases is reported by comparing Figs. 8(a) and (b), and the maximum root mean square error between both fields (from coiled wire and the equivalent lumped model) is about ~2%, within the cross-sectional area shown in Fig. 8 (2mm×2 mm).

![Fig. 7. (a) A PEC coiled wire, and (b) equivalent uniform straight loaded-wire with the same radius but with surface impedance imposed by the distributed inductance per unit length of the coiled wire in (a). Both wires are excited by an $x$-polarized plane wave and the scattered fields are in Fig 8 showing very good agreement.](image)
The metasurface transmission properties for $k_x > k_0$ are analyzed via the analytical model described in this paper and also through full-wave simulations (using CST microwave studio, finite element solver), where we use periodic boundary conditions to emulate the periodicity of the coiled-wires both in the $x$ and $y$-directions. More details on the full-wave simulation setup are provided in the Appendix. We are interested in the near fields generated in the close vicinity of the metasurface along the $z$-direction due to the interaction with the incident evanescent plane wave. We calculate the total average $x$-directed electric field component in the vicinity of the metasurface $\langle E_x(z) \rangle$ as

$$\langle E_x(z) \rangle = \frac{1}{Mpd} \int_{y=0}^{d} \int_{x=0}^{M_p} E_x(x, y, z)e^{-jk_zx}dxdy$$

where $Mpd$ is the area of the metasurface unit cell considered in the full-wave simulations and $M$ is the number of coiled wire periods considered in a single unit cell as detailed in the Appendix. The average electric field $\langle E_x(z) \rangle$ is plotted in Fig. 9(b) varying as a function of $z$ and the frequency of the incident plane wave for $k_z = 2k_0$ (note that $k_z$ changes for each frequency). We note that the total average field $\langle E_x(z) \rangle$ vanishes at some points for $z > 0$, i.e., right on top of the metasurface, see Fig 9(b). To explain such observation and its significance, we resort to a simplified model of the coiled wire metasurface using a homogenized surface impedance $Z_{s,x}$ as in (12). The description of such model is detailed in the Appendix. The evanescent TM' plane wave is incident on such homogenized metasurface, located at $z = 0$ in free space. The reflection coefficient of such plane wave at $z = 0$ is given by

$$\Gamma = \frac{-1}{1 + 2Y_0Z_{s,x}}$$

where $Y_0 = (\omega e_0)/k_z0 = j(\omega e_0)/\sqrt{k_z^2 - k_0^2}$ is the characteristic admittance of the TM' evanescent plane wave. Ignoring conductor losses for simplicity, and assuming that $Z_{s,x}$ is inductive, the term $2Y_0Z_{s,x}$ is real negative and this could cause the reflection coefficient to exceed unity in magnitude, with a $\pi$ phase shift under certain conditions. More precisely, for a lossless inductive metasurface the reflection efficient in (25) assumes a magnitude that is greater than unity (i.e., $|\Gamma| > 1$) only for $k_z > k_0$ when the following condition is fulfilled

$$\frac{\alpha}{2k_z} \left(k_z^2 - k_0^2\right) < \omega e_0 \Im(Z_{s})d < \frac{\alpha}{2k_z} \left(k_z^2 - k_0^2\right) + \sqrt{k_z^2 - k_0^2}$$

(26)

In other words, when (26) is satisfied, the reflected field from an evanescent plane wave excitation of the inductive metasurface is enhanced (i.e., $|\Gamma| > 1$), and consequently this leads to a propagative behavior when multiple metasurfaces are stacked as seen in Section V. Note that the reflected wave
propagates along the +z-direction, and interferes with the incident wave enabling the property that at a specific location, denoted by \( z_{\text{min}} \), the incident and reflected waves would have equal field magnitudes with \( \pi \)-phase difference, thus causing the total field to vanish, for \( k_z > k_0 \) and \( |\Gamma| > 1 \). The location \( z_{\text{min}} \) where the total x-directed field vanishes is calculated easily as

\[
z_{\text{min}} = \frac{-\ln|\Gamma|}{2\sqrt{k_x^2 - k_0^2}}, \quad \text{with} \quad k_x > k_0 \quad \text{and} \quad |\Gamma| > 1 \quad (27)
\]

which is valid only if (26) is satisfied, since in this case \( |\Gamma| > 1 \) and \( \angle \Gamma = \pi \). The location \( z_{\text{min}} \), calculated from (27) using the homogenized model of the metasurface is imposed in the plot in Fig. 9(b), demonstrating good agreement with full-wave simulations. Note that such behavior, with higher-than-unity reflection coefficient, pertains only to inductive metasurfaces for TM\(^e\) evanescent fields. Note also that the field profile in Fig. 9, as well as the location \( z_{\text{min}} \), does not change significantly over the whole considered frequency range, which means that the metasurface is not changing significantly its reactive behavior over the whole frequency spectrum considered in Fig. 9(b).

We further illustrate the average total field profiles for various frequencies and various \( k_z \) in Fig. 10. We compare the full-wave results to those obtained by calculating the scattered field based on the metasurface described by the analytic surface impedance in (8). The scattered field is calculated based on the simple steps summarized in the Appendix based on analysis in [11]. Note that the electric field calculated from the analytical model is continuous at the metasurface location \( z = 0 \); whereas the magnetic field (proportional to the electric field derivative with respect to \( z \)) is discontinuous because of the current induced on the metasurface (see (7)).

We report a good agreement between the full-wave analysis and the analytical model based on (8) in term of the field profile in the close vicinity of the metasurface; as well as the field null points on top of the metasurface. For the cases with larger \( k_z \) (Figs. 10(d)-(f)) the value of \( z_{\text{min}} \) in (27) is very close to the metasurface location as confirmed by full-wave simulations. This indicates that the reflection coefficient in (25) becomes very close to unity in magnitude when increasing \( k_z \) and therefore would limit the enhancement of the reflected waves from the metasurfaces. However, we will show in Section IV that the HM behavior is readily verified for a wide range of frequencies and large transverse wavenumbers.

IV. UNIAXIAL HYPERBOLIC MATERIAL

We consider first a finite number of stacked metasurfaces and investigate the evanescent plane wave interaction with them via full-wave simulations. We show in Fig. 11 the results of plane wave interaction with a 6-layer HM, composed of stacking coiled-wire metasurfaces in free space, separated by a distance \( h \) along the z-direction. The average fields from full-wave simulations are calculated from (24) and plotted in Fig. 11 as well as the corresponding equivalent analytical model using the homogenized surface impedance as explained in the Appendix. We consider incident evanescent plane waves with frequencies of 1 GHz and 5 GHz, with \( k_z = 2k_0 \) in Fig. 11(a) and (b), as well as a case at 3 GHz with \( k_z = 3k_0 \) in Fig. 11(c).

In this multilayered stack, the enhancement of reflected field and the field vanishing locations are also evident; based on the previous observations relative to the single metasurface case in Fig. 10. Furthermore, the multiple coiled wire metasurfaces allow multiple reflections and causing fields inside the layered structure to preserve its envelope along the z-direction, thus allowing for propagation of the impinging evanescent wave into the stack (see Fig. 11(c) as a clear signature of propagative behavior of waves inside the stack). These results confirm the theoretical formulations in Sec. I and II and will be basis to describe the HM as an effective medium that allows propagation of wave with \( k_z \) such that \( |k_z| > k_0 \) (i.e. that would be otherwise evanescent in a homogeneous isotropic space), as we show next.

To obtain a HM at microwaves we are interested in realizing a negative real part of the bulk effective transverse permittivity \( \varepsilon'_t \) for the anisotropic material in Fig. 1. Since in this paper we focus on the uniaxial realization shown in Fig. 3 our goal is to render the real part of the xx-entry of the plane relative permittivity, \( \varepsilon'_t(\omega, k_z) \), negative over a large frequency range and for a large span of transverse wavenumber with \( |k_z| > k_0 \).

The bulk effective relative permittivity in (2) is rewritten here in terms of the metasurface effective impedance \( \varepsilon_x = 1/\sigma_x = R_{x,x} + jX_{x,x} \) as

\[
\varepsilon_x(\omega, k_z) = \left( \varepsilon_d - \frac{1}{\omega \varepsilon_0 h} \frac{X_{x,x}}{|Z_{x,z}|^2} \right) - j \frac{1}{\omega \varepsilon_0 h} \frac{R_{x,x}}{|Z_{x,z}|^2}. \quad (28)
\]

Assuming TM\(^e\) polarized waves with electric field polarized along \( x \), i.e., with wavevector is in the \( x-z \) plane, the dispersion relation (4) reduces to

\[
\frac{k_x^2}{\varepsilon_x} + \frac{k_z^2}{\varepsilon_z} = k_0^2, \quad (29)
\]

and the hyperbolic-like dispersion is realized when \( \varepsilon'_x < 0 \) for \( |k_z| > k_0 \), that is one of the main goals of this paper (here we left the arbitrary host permittivity \( \varepsilon_d \) for generality). Based on (28), the condition \( \varepsilon'_x < 0 \) is equivalently rewritten as

\[
\frac{X_{x,z}}{|Z_{x,z}|} > \varepsilon_0 \varepsilon_d \omega h \quad (30)
\]

which means that the metasurface inductive admittance is larger than the capacitive admittance representing the electric field stored in the dielectric spacer of thickness \( h \) between two adjacent metasurfaces. Note that because of the purposely introduced inductive loading discussed in Section III, \( X_{x,z}(\omega, k_z) \) is positive also for large \( |k_z|/k_0 \) values. Using the metasurface effective surface impedance (12) that is a good approximation of (8) when (11) is satisfied, the condition (30)
for negative effective transverse permittivity $\varepsilon'_t < 0$ is rewritten as

$$
\frac{\text{Im}(Z_{\text{PEC}}^0)\left(1 - \frac{k^2}{k_0^2}\right) + \text{Im}(Z_id)}{Z_{\text{PEC}}^0\left(1 - \frac{k^2}{k_0^2}\right) + Z_id^2} > \varepsilon_0 d \omega h
$$

(31)

where we note that $\text{Im}(Z_{\text{PEC}}^0) = \frac{1}{2}\eta\alpha_{ABC} > 0$ for $r_0 < d/(2\pi)$. Solving (31) for $\text{Im}(Z_i)$ leads to the following inequalities that must be satisfied by $\text{Im}(Z_i)$ to provide $\varepsilon'_t < 0$:

$$
\left(\varepsilon_0 d \omega h - \text{Im}(Z_{\text{PEC}}^0)\right)\left(1 - \frac{k^2}{k_0^2}\right) + \text{Im}(Z_id) > 0
$$

(32)

Fig. 10. Comparison between fields evaluated via full-wave simulations for a single metasurface made of an array of coiled wires and those obtained via the analytical model based on a metasurface with homogenized surface impedance $Z_{\text{hx}}$ in (8). The metasurface is excited by an evanescent plane wave, at three different frequencies and two transverse wavenumbers $k_x$ larger than $k_0$ as seen in the inset. The vertical dashed lines denote the location of the metasurfaces ($z = 0$). The agreement is excellent. Note also the location $z_{\text{min}}$ where the total field vanishes because of the reflection coefficient larger than unity.

Fig. 11. Similar to Fig. 10 but considering an evanescent wave impinging on a stack of six metasurfaces with period $h = 0.82$ mm, for three different cases of evanescent incident plane wave frequency and transverse wavenumber $k_x$ as seen in the inset (in all cases $k_x > 0$). The vertical dashed lines denote the location of the metasurfaces, with the top one located at $z = 0$. Blue line = field evaluated via analytical model developed in this paper; red circles = full-wave simulations. The agreement is excellent.
\( k_x \) span for which \( \epsilon'_x < 0 \) is achieved. In other words, propagation in a metamaterial as in Fig. 1 of a plane wave with a certain transverse wavenumber \( |k_x| > k \) is inhibited by any amount of losses that exceed a given value provided by any of the two following inequalities:

\[
\text{Re}\left( \frac{Z_l d}{Z_c} \right)^2 \geq 1 - \left[ 1 - \text{Im}\left( \frac{Z_l d}{Z_c} \right) - \frac{1}{2} \eta \alpha_{AB} \left( 1 - \frac{k_x^2}{k^2} \right) \right]^2 \quad (33a)
\]

\[
\text{Re}\left( \frac{Z_l d}{Z_c} \right) \geq \frac{1}{2} \eta \alpha_{AB} \left( \frac{1 - k_x^2}{k^2} \right) \quad (33b)
\]

Assuming the ideal case of a lossless conductors, one has \( \text{Re}(Z_l) = 0 \), and the inequality in (32) reduces to

\[
0 < \text{Im}(Z_l d) + \frac{1}{2} \eta \alpha_{AB} \left( 1 - \frac{k_x^2}{k^2} \right) < \frac{1}{\epsilon_0 \epsilon'_d \omega h} \quad (34)
\]

The greater-than-zero side of (34) indicates that one requires an overall inductive metasurface, in other words \( X_{s,z} = \text{Im}(Z_{s,z}) > 0 \), to realize \( \epsilon'_x < 0 \). Moreover the inequality on the right hand side of (34) is rewritten as \( X_{s,z} < 1/(\epsilon_0 \epsilon'_d \omega h) \), which is equivalent to the original condition (30) when losses are negligible. This condition imposes a limit on the large value of the metasurface inductive impedance \( X_{s,z} \) since it must be considered in “parallel” with the equivalent capacitive load represented by the dielectric layers.

We now show the condition on the wire inductive load \( X_l = \omega L_l \) [see (16)] to ensure that a HM has \( \epsilon'_x(\omega, k_x) < 0 \) at a given \( k_x \). In the case of negligible losses this condition is represented by the left inequality in (34) that leads to

\[
L_l > L_{l,min}, \quad \text{with} \quad L_{l,min} = \left[ \frac{k_x^2}{k} \right]^2 - \frac{1}{2\pi c} \ln \frac{d}{2\pi r_0}. \quad (35)
\]

As discussed above, the right inequality in (34) represents the maximum loading \( X_{s,z} < 1/(\epsilon_0 \epsilon'_d \omega h) \) to be sure \( \epsilon'_x(\omega, k_x) < 0 \). Solving it for the loading inductance leads to

\[
L_l < L_{l,min} + \Delta L, \quad \text{with} \quad \Delta L = \frac{1}{\epsilon_0 \epsilon'_d \omega h}. \quad (36)
\]

This means that the wire inductance per unit length \( L_l \) whose expression is given in (17) must be within a range dictated by (31) and (32), i.e., within a range \( \Delta L \) that can be rewritten also as \( \Delta L = \mu_0 L \left( 4\pi^2 d \epsilon'_d h \right) \). Note that this range decreases with increasing frequency and to keep \( \Delta L \) large enough to allow values from (17) to fall within the useful range it is advantageous to reduce the wires’ distance \( d \) and the metasurface separation \( h \) with respect to the wavelength.

It is important to note that we seek to realize hyperbolic dispersion for a wide range of \( k_x \) values, possibly even starting at \( |k_x| \approx k \), which simply means that \( L_l < \Delta L \). On the other hand requiring that \( \epsilon'_x(\omega, k_x) < 0 \) for large values of \( \frac{|k_x|}{k} \), (35) implies that the value \( L_l \) must be very large. Therefore the only way to ensure that \( \epsilon'_x(\omega, k_x) < 0 \) for a large interval of \( k_x \) values is to have a large \( \Delta L \), since the coil radius \( r_0 \) cannot be reduced indefinitely due to the minimum inductance value \( L_{l,min} \). Furthermore, the frequency dependency of \( \Delta L \) implies the existence of a cutoff frequency above which \( \epsilon'_x(\omega, k_x) < 0 \).

V. EFFECTIVE MEDIUM APPROXIMATION

In the aforementioned cases, we observe that the \( x \)-component of the electric field inside the multilayered HM has constant amplitude, which means that HM allows such evanescent wave to propagate inside the multilayered HM. Indeed, according to the homogenization formula (3) the HM made of such a stack of metasurfaces supports propagative spectrum for \( k_x > k_0 \) as discussed next.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig12}
\caption{(a) Real and (b) imaginary parts of the wavenumber \( k_x \) varying \( k_x \) which represents the iso-frequency dispersion diagram for TM wave with hyperbolic dispersion for various frequencies in the HM. The plots in (a) and (b) are calculated using formula (4) utilizing effective transverse relative permittivity in (2) that is based on the coiled-wire array inductive metasurface effective surface impedance (8) taking into account the summation \( A \) with \( N=1000 \) terms and the coiled-wire parameters for his particular example are reported in Fig. 4. We only plot branches of \( k_x \) for which \( \text{Re}(k_x) \text{Im}(k_x) > 0 \) in the evanescent spectrum \( k_x > k_0 \) that represent backward waves.}
\end{figure}

A. HYPERBOLIC WAVEVECTOR DISPERSION OF A COILED-WIRE-ARRAY-BASED HM

In this section, we provide an example showing the HM dispersion behavior based on the homogenized dielectric permittivity \( \epsilon_x(\omega, k_x) \) along the \( x \)-direction of an artificial medium made of a stack of alternating dielectric spacers and inductive metasurfaces as described in Sections III and IV. Restricting the analysis to an incident TM wave with \( E \) field in the \( (x,z) \) plane, the equivalent dielectric permittivity of the stack along the \( x \)-direction, \( \epsilon_x(\omega, k_x) \), is estimated at first order using (2), where \( \epsilon_x \) is restricted to \( \epsilon_x \). In (2) the surface conductivity \( \sigma_s = 1/Z_{sb} \) is estimated using (8) with the distributed load \( Z_l \) from (16), (17) and (18) and the data from Fig. 4.
We calculate the surface impedance $Z_{s,x}$ (12) when considering the infinite summation $\Lambda$ (which is approximated for numerical calculations by truncating the series in (9) with $N=1000$ terms) and thus we consider the parameter $\Lambda$ in the surface impedance evaluation and consequently in calculating $\varepsilon_x$ in (38). We also consider ohmic losses in the wires based on the conductivity of copper. As such, using the wavevector dispersion equation of the homogenized HM (4), we plot the isofrequency wavevector dispersion for the HM made of a coiled wire array in Fig. 12 showing both real and imaginary parts of the wavenumber $k_i$ calculated using (4). The wavevector dispersion portrays a propagative behavior for $k_i > \sqrt{\varepsilon_d k_0}$ with a large value of $\text{Re}(k_i)$ while the imaginary part $\text{Im}(k_i)$ is relatively smaller for the range $\sqrt{\varepsilon_d k_0} < k_i < 15k_0$ for the three different frequencies reported in Fig. 12. The limit for obtaining this HM behavior are discussed next as well as the approximations to obtain the effective permittivity.

B. Limits for HM behavior: $\text{Re}(\varepsilon_x(\omega,k_x)) = 0$ and the importance of the infinite sum term in the surface impedance expression (8)

We first compare in Fig. 13 a metasurface surface impedance (8) with the one obtained when neglecting the infinite summation $\Lambda$ in (10) and thus we approximates $\alpha$ by $\alpha_{ABC}$ in (9).

In particular, Fig. 13 shows the relative error made using the analytical approach based on neglecting $\Lambda$ in (9) instead of the numerical one that includes $\Lambda$ from (10) on the real part of the surface conductivity with actual copper losses. In particular, we note that on the range $k_x/k \leq 10$ and $f \leq 8.5\text{GHz}$, the error is lower than 10%.

We investigate now the frequency range and wavenumber span for which HM behavior is guaranteed. Based on the data in Fig. 4, with $h = 1$ mm and $d = 1$ mm, Fig. 14 gives the estimated frequency-wavenumber range that guarantees the real part of $\varepsilon_x$ to be negative (hence HM behavior is found). The value of $\varepsilon_x$ is calculated based on (28), numerically approximating the infinite sum in (10) with $N=1000$ terms. The boundary defined by $\text{Re}(\varepsilon_x(\omega,k_x)) = 0$ is evaluated with the sum in (10), and compared with the limit based on the inequality in (31) that neglects the term $\Lambda$ in (9). HM behavior is found over a large frequency range and over a large $k_x$-wavenumber span. In particular, the range $|k_x|/k \leq 10$ and $f \leq 8.5\text{GHz}$ experiences $\text{Re}(\varepsilon_x(\omega,k_x)) < 0$. Furthermore, on this range, analytical (neglecting $\Lambda$) and numerical (considering $\Lambda$) approaches lead to very similar $\text{Re}(\varepsilon_x(\omega,k_x)) = 0$ limits.

HM behavior is guaranteed below the solid white line for any frequency till $|k_x|/k \leq 23$. For $|k_x|/k > 23$, HM behavior is guaranteed in the frequency interval between the two solid white lines, and such frequency interval decreases with increasing $|k_x|$.

Note that for $|k_x|/k > 10$ the limit $\text{Re}(\varepsilon_x(\omega,k_x)) = 0$ is not estimated correctly by the analytical approach (i.e., by neglecting $\Lambda$). However assumptions of smallness of geometric dimensions such as $d << 2\pi/k_x$ are also less and less valid as the transverse wavenumber increases and so is the homogeneous model in (2). Above $|k_x|/k > 10$, the frequency range of validity of the approach presented in this paper shall be reduced as the transverse wavenumber ratio $|k_x|/k$ increases.
contours on which \( \varepsilon'_x = 0 \) considering the summation \( \Lambda \) while the dotted black lines represent the same when neglecting the summation \( \Lambda \) in the calculations. Note that the black dotted lines represent the limits of the inequality in (31).

### C. Impact of high loss

Fig. 15 presents the effect of ohmic losses of the coiled wires on the threshold of the real part of the effective bulk permittivity, i.e., for a given amount of loss we show the \( \varepsilon'_x(\omega, k_x) = 0 \) curve. We recall that HM behavior is ensured in the domain enclosed by these contours. As in Fig. 14, solid lines are obtained using the numerical approach (thus approximating the infinite summation \( \Lambda \) with \( N=1000 \) terms), and dots are obtained using the analytical one (thus neglecting the infinite summation \( \Lambda \)).

Based on the coiled wire parameters in Fig.4, with \( h = 1 \text{ mm} \) and \( d = 1 \text{ mm} \) and considering actual copper losses, \( \text{Re}(Z_i) d << Z_c \) in the considered frequency range, thus the \( S \) parameter defined in (32) is small compared to \( Z_c \). Condition (32) tends to condition (34), which means that the effect of Ohmic losses is negligible compared to the inductance per unit length, in the considered frequency range. Therefore, we expect the hyperbolic dispersion not to be significantly affected by actual copper losses in our numerical example. This is numerically confirmed in Fig.15. The limit above which the HM behavior is no longer observable when considering losses is much higher than the actual copper losses. With the parameters chosen in Fig. 4, the effect of losses starts appearing for a volume resistivity that is three orders of magnitude the copper resistivity \( \rho = 10^3 \rho_{Cu} \). Since the resistance \( R_l \) of the equivalent distributed load depends on the square root of the volume resistivity (19), it means that the HM behavior limit starts to be affected for several dozens of time the \( R_l \) induced by the coiled copper wire. For \( R_l \) increased by four orders of magnitude (green curve), the frequency span reduces from 8.5 GHz to 6.7 GHz.

![Fig. 15. Contours of \( \varepsilon'_x = 0 \) varying as a function of \( k_x \) and frequency. HM behavior is guaranteed below the top curves and above the bottom ones. The lossless case is not shown here since it basically coincides with the curves for the case with \( \rho = \rho_{Cu} \). Solid lines are obtained by considering the infinite sum in (10) to evaluate \( \Lambda \); dotted lines are obtained by neglecting the effect of \( \Lambda \) in (9).](image)

### VI. Conclusion

We have proposed and explored a novel approach to realize a “hyperbolic metamaterial” (HM) at microwave frequencies. In the HM, wave spectrum that would be otherwise evanescent in free space is able to propagate. We have demonstrated that, by stacking inductive metasurface made of coiled-wire arrays, a homogeneous transverse permittivity with negative real part is obtained which in turns generates a hyperbolic isofrequency wavevector dispersion for large transverse wavenumber. We have shown that metasurfaces made of coiled-wire arrays show an inductive behavior for large transverse wavenumbers, larger than the free space wavenumber, based on an analytical approach verified by full-wave simulations. We have also derived formulas describing the HM effective medium properties, as well as the limitations for the propagative frequency and wavenumber spectra in the HM taking into account parasitic capacitances and ohmic losses. With the realistic parameters chosen in the provided examples, the hyperbolic wavevector dispersion behavior is obtained over the frequency range up to \( \sim 8.5 \text{ GHz} \) and remarkably for \( |k_x|/k_0 \) values up to several units.

An important aspect is that there is theoretically a large margin for ohmic loss not to affect the negative effective transverse permeability of the HM. Thus, losses represented by the effective transverse epsilon could be manipulated and further engineered depending on the HM application.

The uniaxial nature of the metasurfaces in this paper restricts the hyperbolic properties to TM\(^t\) waves polarized along the \( x \)-direction (the direction of the coiled wires). However the same approach can be adopted to analyze hyperbolic materials for arbitrarily polarized TM\(^t\) waves, using stacks of metasurfaces made of two-dimensional periodic planar grids, like crossed coiled wires.

### APPENDIX

We summarize here the setup used to perform full-wave simulations in Section IV of the coiled wires based metasurface and to show the inductive behavior for large wavenumbers. Consider a single metasurface made of an array of coiled wires as in Fig. 3, whose parameters are provided in Fig. 4. We use periodic boundary conditions in the finite element methods (implemented in frequency domain solver in CST Microwave Studio) to simulate the interaction of a plane wave excitation with a metasurface with unit cell dimensions of \( M_p \) and \( d \) in the \( x \)-and \( y \)-directions, respectively. We take the coil pitch \( p = 76 \mu m \) as in Fig. 4, and so we consider an array period \( d \) such that \( d = M_p = 1.064 \text{ mm} \), where \( M = 14 \) periods of the coiled wire. The full-wave simulation is carried out considering a square periodic unit cell with size \( M_p \times d \). In Fig. A1 a schematic representation of the simulation setup used to produce plots in Fig. 10 and Fig. 11 is depicted. Note that our simulation requires an evanescent plane wave excitation from free space above the metasurface. This is achieved in the setup in Fig. A1 that emulates total internal reflection for a plane wave propagating at an angle \( \theta^t \) in the background medium (\( z > z_c \)) with relative permittivity \( \varepsilon_r > 1 \), and the transmitted evanescent wave (in the free space with \( z < z_c \)) is used to excite the metasurface at \( z = 0 \).
Indeed, in order to produce a TM\textsuperscript{e} evanescent wave for \(z < z_r\) with a specific given value \(k_x\) and frequency such that

\[
\begin{cases}
|k_x| > k_0, & \text{Evanescent for } z < z_r, \\
|k_x| < \sqrt{\varepsilon_b k_0}, & \text{Propagating for } z > z_r,
\end{cases}
\] (A1)

the relative permittivity of the background medium \((z > z_r)\) is then set in the full-wave setup to realize the required value of \(k_x\) and therefore the evanescent decay for \(z < z_r\), as

\[
\varepsilon_b = \left( \frac{k_x}{k_0 \sin \theta^i} \right)^2
\] (A2)

where \(\theta^i\) is the incident angle (real valued) of the propagating plane wave in the background medium \((z > z_r)\). The angle \(\theta^i\) is taken for simplicity as 45°. In the full-wave simulations, \(z_r\) is assumed to be 10.59 mm. Accordingly, the relative permittivity of the background medium \((z > z_r)\) is chosen to be \(\varepsilon_b = 8\) to achieve \(k_x = 2k_0\), and \(\varepsilon_b = 50\) to achieve \(k_x = 5k_0\). The transmitted field is then evaluated at \(z = H = 30\) mm (see Fig. A1).

The equivalent Transmission Line (TL) network used in some of the calculations is also depicted in Fig. A1 for the case of the single metasurface excited by a plane wave. The analysis of the TL network is carried out using the transfer matrix method [11]. The lumped inductive shunt load used in the loaded TL network is represented by the surface impedance \(Z_{x,x}(\omega) = 1/\sigma_{x,x}\) given in (8). However, in the cases in Section III.E and IV we have ignored ohmic losses (i.e. \(R = 0\) which means the coiled wire are made of PEC). The characteristic impedances of the TLs for TM\textsuperscript{e} waves in free space is \(Z_0 = 1/Y_0 = k_{cz}/(\omega \varepsilon_b)\), whereas in the background medium it is \(Z_b = k_{cz}/(\omega \varepsilon_b \varepsilon_b)\), where \(k_{cz} = -\sqrt{k_x^2 - k_0^2}\) and \(k_{cz}^2 = \sqrt{\varepsilon_b k_0^2 - k_x^2}\), and the square roots are evaluated via their principle root to produce a number with positive real part.

Fig. A1. Loaded transmission line model and the full-wave setup used in Section VI to calculate the near fields near the coiled wires metasurface excited by an evanescent plane wave. The reference plane for the evanescent plane wave is at \(z = z_r\) at which the incident field \(x\)-component is taken to be equal unity.

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