Degenerate band edge laser

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We propose a class of lasers based on a fourth-order exceptional point of degeneracy (EPD) referred to as the degenerate band edge (DBE). EPDs have been found in parity-time-symmetric photonic structures that require loss and/or gain; here we show that the DBE is a different kind of EPD since it occurs in periodic structures that are lossless and gainless. Because of this property, a small level of gain is sufficient to induce single-frequency lasing based on a synchronous operation of four degenerate Floquet-Bloch eigenwaves. This lasing scheme constitutes a light-matter interaction mechanism that leads also to a unique scaling law of the laser threshold with the inverse of the fifth power of the laser-cavity length. The DBE laser has the lowest lasing threshold in comparison to a regular band edge laser and to a conventional laser in cavities with the same loaded quality ($Q$) factor and length. In particular, even without mirror reflectors the DBE laser exhibits a lasing threshold which is an order of magnitude lower than that of a uniform cavity laser of the same length and with very high mirror reflectivity. Importantly, this novel DBE lasing regime enforces mode selectivity and coherent single-frequency operation even for pumping rates well beyond the lasing threshold, in contrast to the multifrequency nature of conventional uniform cavity lasers.

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I. INTRODUCTION

Demonstration of a low-threshold laser operating at a single frequency is an important quest in the optical and physical sciences. In this regard, the use of periodic structures with engineered dispersion diagram is a popular and effective way to enhance the interaction between the gain medium and the electromagnetic wave therefore tailoring the lasing characteristics of active structures. In the last decades, photonic-crystal-based optical devices and distributed feedback (DFB) lasers have demonstrated inimitable features and high performance due to their unprecedented dispersion characteristics, high quality ($Q$) factors, and field enhancement properties [1–6]. In particular, increasing the $Q$ factor of photonic-crystal-based cavities results in a significant reduction of the lasing threshold [2–4,7–11]. Therefore, many techniques have been proposed to enhance the $Q$ factor of photonic-crystal-based cavities for engineering light sources, such as introducing a small disorder or a defect into the crystal [7,12], using photonic-crystal heterostructures [9,13], and by locally modulating the width of the photonic-crystal waveguide [14,15]. Recent advances in developing optical lasers rely on engineering the response of the cavity structures by employing plasmonic nanocavities [16–18], photonic band edges [19–25], parity-time (PT) symmetry breaking [26–32], or by exploiting unique structural topologies including metamaterials [33–36] and metasurfaces [37,38].

In this paper, we propose a class of single-frequency lasers made of a cavity with degeneracy of four Floquet-Bloch eigenwaves coherently interacting with an active medium. Such a degeneracy is found in periodic structures whose dispersion relations develop points of degeneracy at which state eigenvectors representing Floquet-Bloch eigenwaves coalesce [21,22,39]. Those points in the spectrum of the “cold” periodic structure (“cold” refers to the absence of the gain media) are associated with a regular band edge (RBE), a stationary inflection point (SIP) [40,41], or a degenerate band edge (DBE) [22,23], resulting in a second-, third-, or fourth-order degeneracy of Floquet-Bloch eigenwaves (in both eigenvalues and eigenvectors), respectively. We refer to such points as exceptional points of degeneracy (EPDs).

EPDs have been commonly associated with the presence of gain and/or loss and often related to parity-time (PT)symmetry [26–32]. However here we point out that the EPD may be induced in electrodynamical systems also in absence of gain and losses. In gain and loss balanced systems (like in systems with PT symmetry), EPDs occur in the parameter space of the system described by the evolution of their eigenmodes either in time (for coupled resonators such as those in [31,32,41]) or in space (for coupled waveguides such as those in [28,42–44]). On the other hand, EPDs are also realizable in spatially periodic media supporting Floquet-Bloch waves such as photonic crystals and periodic waveguides exhibiting RBEs, SIPs, and DBEs, in absence of gain and loss. It is then important to emphasize that indeed both cases of (i) gain and loss induced EPDs and (ii) periodicity induced EPDs follow the same mathematical fundamental theory of degenerate operators (see pages 63–67 in [45]), i.e., the eigenvalues and eigenvectors characteristics in both cases lead to a Jordan block degeneracy as described in [22,43–45]. In this paper we investigate the space evolution of guided eigenwave in periodic waveguides (i.e., along the $z$ direction) at and near EPDs, that would result in unique lasing features.
In particular we focus on the DBE degeneracy \([22,39,46,47]\) which arises when four Floquet-Bloch eigenvectors coalesce in spatially periodic structures supporting multiple polarization states that are periodically mixed. It features a frozen-wave resonance relying on a fundamental property of EPDs that causes eigenwave solutions inside a periodic structure to diverge, leading to giant field enhancement \([22,47]\). However, it is important to stress that the DBE, which is a fourth-order EPD, occurs in a passive and lossless system, i.e., without the need of gain or loss. Some DBE characteristics have been shown to occur at optical frequencies using perturbed coupled silicon waveguides \([48,49]\), or a chain of ring resonators coupled to a waveguide \([50]\), as well as in metallic waveguides at microwaves \([39]\). There have been also significant efforts in analysis, design, and experimental realization of the DBE structures and its slow-wave properties at both microwave \([51–53]\) and optical frequencies \([54,55]\). It is important to point out that even though DBE is a precise EPD condition occurring in lossless waveguides, experimental studies made both at microwaves \([53]\) and optical frequencies \([50,51]\) have confirmed the existence of features associated with the DBE in the presence of losses and fabrication tolerances. Indeed, the robustness of DBE features against perturbations due to possible fabrication tolerances that may arise during fabrication was demonstrated for DBE coupled resonator optical waveguides (CROWs) in \([50]\). The DBE has led to observing giant gains in optical cavities \([47]\), however here we leverage a general EPD concept to propose the regime of lasing, resulting in low-threshold and single frequency operation of the degenerate band edge laser.

In previous work \([47]\), an analysis of an ideal multilayer anisotropic medium with ideal gain (not represented with rate equations in a multilevel energy setup as done here) was carried out using the transfer-matrix method which led to possible gain enhancement in cavities with DBE. The analysis in \([47]\) has resulted in an oscillation threshold of such DBE active cavities that scales as \(N^{-5}\), where \(N\) is the number of unit cells as seen from Fig. 1. In this paper we carry a comprehensive analysis of the proposed DBE laser: (i) We demonstrate a special feature of lasing-mode selection in the DBE laser that leads to a single frequency operation (the analysis of the lasing threshold is elaborated in Secs. II–IV). (ii) The study included time domain simulations of electromagnetic fields and evolution of rate equations describing gain arising from a multilevel energy system. (iii) We further show that the proposed DBE laser features a significantly lower lasing threshold compared to a conventional RBE laser \([24]\) and a uniform Fabry-Perot cavity (FPC) while having the same gain medium, length, and same total loaded \(Q\) factor [see Fig. 1(a)]. (iv) The low lasing threshold of the proposed DBE laser, which can be up to two orders of magnitude lower threshold than conventional lasing cavities as shown in Fig. 1(a), is ascribed to the enhanced interaction between the gain media and the cavity featured by all four degenerate Floquet-Bloch eigenvectors at the DBE wavelength as will be explained throughout the paper. (v) We also demonstrate that the DBE laser operates without the need of cavities mirrors and the threshold is independent of mirror reflectivity as shown in Sec. IV. These findings are especially valuable for further developing optical devices based on degeneracy properties. Note that although the DBE is a slow-light phenomenon occurring strictly in a lossless periodic waveguide, we propose the DBE lasing regime in a fully coupled system composed of a cold DBE cavity and a nonlinear gain medium [modeled by \((4)\) in Sec. II B] for which the interaction is investigated using time-dependent nonlinear evolution equations.

The concepts discussed are based on an interaction regime between light and active material. Typically, laser sources operate based on conventional Fabry-Perot cavity resonances and that causes challenges especially in the semiconductor laser realm. However, here we employ a fourth-order eigenwave...
degeneracy (i.e., they form a single degenerate eigenmode) to enforce lasing mode selectivity and conceive a class of low-threshold lasers whose threshold is independent of mirror reflectivity.

The layout of the paper is organized as follows. First, we show possible implementations of cold coupled optical waveguides, whose Floquet-Bloch eigenvalues support the DBE, in Sec. II A. We then theoretically investigate the DBE laser based on evolution equations for waves in coupled waveguides that account for spatial periodicity as well as loss and gain in Sec. II B and Appendix A. The properties of the cold DBE structure as well as the steady-state gain medium response are introduced and investigated in Secs. II C and II D, respectively. We then report the evolution of the lasing action inside the DBE laser using the finite difference time domain (FDTD) algorithm as well as the lasing threshold analysis [in Fig. 1(a)] in Sec. III and Appendix B. Finally, we demonstrate the effect of losses on the loaded Q factor and provide comparisons between conventional lasers and the proposed DBE laser in terms of lasing threshold in Sec. IV.

II. LASER THEORY IN COUPLED WAVEGUIDES WITH EPDs

In this paper, we propose and investigate a different class of EPD lasers. Our proposed laser operates near the DBE which guarantees coherent single frequency oscillation as well as low threshold.

A. Pair of periodic coupled waveguides with four degenerate eigenwaves

Among the many possible lossless optical coupled waveguide geometries that may exhibit DBE, i.e., an EPD caused by coalescence of four Floquet-Bloch eigenvalues into one degenerate eigenvalue, we illustrate in Figs. 1(b) and 1(c) two representative waveguide examples. The dispersion relation with DBE is shown in Fig. 2. The optical waveguide in Fig. 1(b) is composed of periodic segments of coupled and uncoupled optical waveguides, with period d. An alternative coupling mechanism could be also realized through optical resonators as couplers to the waveguide [as in Fig. 1(c)]. The reason for utilizing optical resonators (such as microrings or microdisks [56–58]) is their ubiquitous use in lasers due to their high loaded Q factor and ease of fabrication. In the example shown in Fig. 1(c) the DBE cavity is made of waveguides coupled to a chain of coupled resonator optical waveguides (CROWs) as shown in [50]. (The conventional CROW topology would support only an RBE [59–62].) Furthermore, the DBE CROW proposed in [50] was shown to exhibit remarkably high Q-factor resonances near the DBE that are robust against disorders and perturbations. Note that other geometries can also be designed and implemented using various semiconductor photonics technologies [46]. The geometries shown in Figs. 1(b) and 1(c) are simply represented using their equivalent coupled waveguide model in Fig. 1(d), constituting WG1 and WG2, that is the model used in the rest of the paper. Such periodic waveguides have a dispersion diagram as in Fig. 2, exhibiting the DBE at \( f_d = \omega_d / (2\pi) \approx 193.4 \) THz, i.e., at \( \lambda_d = 1550 \) nm, using a pair of periodic coupled waveguides as in Fig. 1(d) with the parameters in Appendix C. Note that the dispersion relation in the vicinity of DBE frequency is approximated by \( (\omega_d - \omega) \propto h(k - k_d)^4 \) where \( k_d = \pi / d \) is the wave number at the DBE, and \( h \) is a geometry dependent parameter. At any \( \omega \) the Floquet-Bloch eigenvalues in a pair of coupled periodic waveguides comprise four \( k \) wave numbers, associated to four eigenvectors that are, in general, mutually independent. However, at a DBE frequency the four Floquet-Bloch eigenvalues coalesce, in wave number and eigenvectors. A consequence of Floquet-Bloch eigenvalue degeneracy is that independent basis of wave propagating must come in the form of generalized eigenvectors at the DBE (see Chap. 7.8 in [63] as well as [64,65]). At the DBE, waves associated to those generalized eigenvectors grow linearly, quadratically, and cubically with the \( z \) coordinate, having vanishing group velocity yet still satisfying Maxwell’s equations [64,66,67]. In addition, finite structures made of such periodic coupled waveguide experience unusual FPC resonances compared to uniform FPCs [47]. The DBE laser proposed here oscillates based on the FPC resonance closest to the DBE frequency and therefore enables a remarkable low-threshold single frequency lasing as will be shown in Secs. III and IV.

Note that the concepts exposed in this paper are general and applicable to any waveguide geometry exhibiting a DBE. To illustrate the DBE laser concept however we specifically refer to an illustrative example of periodic waveguides modeled as a cascade of coupled and uncoupled waveguides as in Fig. 3 (details in the appendixes) with coupled-wave equations described as follows.

B. Time domain formulation of the DBE laser action

We describe in detail the theory of coupled waves propagating (along the \( z \) direction) in coupled waveguides with EPDs.
We assume that each of the two waveguides has only a single propagation eigenwave (in each of the \( \pm z \) directions), so the coupled waveguide system has two propagating eigenwaves (in each direction). Here coupling between the two waveguides (WG1 and WG2) is mediated phenomenologically through coupled differential wave equations [as in (1)] involving wave amplitudes. (Note that this approach is known in the radio frequency as coupled transmission lines [68] and it is inherently related to the conventional coupled-mode theory [69–74] widely used for optical systems.) We again point out that the lasing regime proposed here is based on a fully coupled system modeled by nonlinear evolution equations in time, describing the dynamics of electromagnetic waves in the DBE cavity that incorporates gain material. Discussions about gain enhancement properties for slow-light waveguides can be found elsewhere, for instance in [24,47]. We stress that, strictly speaking, the presence of gain or loss detunes the system away from the mathematical DBE condition as we have previously investigated in other cases [75–76,47]. However, important field properties of the special DBE degeneracy are retained when gain is not high. In addition, our evolution equations correctly take into account gain in a slow-light system and would provide proper results also for large gain.

We consider two traveling waves, in WG1 and WG2, that are described by their spatiotemporal amplitudes \( a^+_1(z,t) \) and \( a^+_2(z,t) \). The two waves propagate along the coupled waveguide in the +z direction. Their amplitudes are normalized in such a way as to represent power waves as pointed out in [77], so that \( S^+(z,t) = (a^+_1(z,t))^2 + (a^+_2(z,t))^2 \) is the instantaneous power flux in the coupled waveguides along the +z direction. As such, the wave amplitudes are conveniently expressed using a two-dimensional vector \( \mathbf{a}^+(z,t) = [a^+_1(z,t)\ a^+_2(z,t)]^T \), where superscript \( T \) stands for transpose. From here onward, bold symbols denote vectors, while bold double-underlined symbols denote \( 2 \times 2 \) matrices and bold single-underlined symbols denote \( 4 \times 4 \) matrices.

The space-time evolution equations for the wave amplitudes \( \mathbf{a}^+(z,t) \) in a uniform, lossless coupled waveguide are given by

\[
\frac{\partial}{\partial z} \begin{bmatrix} a^+_1(z,t) \\ a^+_2(z,t) \end{bmatrix} = \mathbf{n} \frac{\partial}{\partial t} \begin{bmatrix} a^+_1(z,t) \\ a^+_2(z,t) \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix},
\]

where \( c \) is the speed of light in vacuum, \( n_{11}, n_{22}, \) and \( n_{m} \) are effective refractive indices of the coupled waveguide, respectively, and \( \mathbf{n} \) is a \( 2 \times 2 \) matrix whose eigenvalues are the effective refractive indexes of the propagating waves in the coupled waveguides. Note that the coupling between the two waveguides is present through the off-diagonal entries \( n_m \) of the matrix \( \mathbf{n} \). Indeed, when \( n_m = 0 \) the two waveguides are uncoupled and the space-time evolution of the waves in the system described by (1) turns into two uncoupled equations whose solutions are the natural propagating eigenwaves in WG1 and WG2. When WG1 and WG2 are coupled, the two eigenwave solutions are found by solving the coupled-wave equation (1). Thanks to reciprocity, the eigensolutions of (1) obey the symmetry \( t \rightarrow -t \). Therefore, independent eigenwaves may also propagate in the negative \( z \) direction, and their amplitudes are denoted by a two-dimensional vector \( \mathbf{a}^-(z,t) = [a^-_1(z,t)\ a^-_2(z,t)]^T \).

\[
\frac{\partial \mathbf{a}^+(z,t)}{\partial z} = -\mathbf{n} \frac{\partial \mathbf{a}^+(z,t)}{\partial t}; \quad \frac{\partial \mathbf{a}^-(z,t)}{\partial z} = \mathbf{n} \frac{\partial \mathbf{a}^-(z,t)}{\partial t}.
\]

(2)

In (1), that applies to a lossless system, the matrix \( \mathbf{n} \) is purely real and symmetric (so it is Hermitian). The entries of \( \mathbf{n} \) are real valued and are associated to the refractive index \( \hat{n} \) of the coupled waveguide system. The matrix \( \mathbf{n} \) appears as a simple multiplier because in a frequency domain description we neglect material and waveguide frequency dispersion. This is a valid approximation since we investigate lasing action in a narrow frequency range given by the emission spectrum of the active atoms. In general, one may construct a first-order evolution equation in the waveguide by assuming a state vector \( \Psi(z,t) \) comprising the wave amplitudes, that evolves in the lossless coupled waveguide as

\[
\frac{\partial \Psi(z,t)}{\partial z} = -\tilde{\mathbf{M}} \frac{\partial \Psi(z,t)}{\partial t}, \quad \Psi(z,t) = \begin{bmatrix} \mathbf{a}^+(z,t) \\ \mathbf{a}^-(z,t) \end{bmatrix},
\]

where the \( 4 \times 4 \) matrix \( \tilde{\mathbf{M}} \) is a block-diagonal system matrix comprised of the matrix \( \mathbf{n} \) for each uniform waveguide segment, as given in Appendix A.

It is important to point out that the coupling mechanism and the transfer of power between the coupled waveguides are well understood from the evolution of the coupled waves as described above. The aforementioned analysis resembles the coupled-mode theory for optical waveguides [69–74]. However, we recall that our approach is in fact equivalent to transmission line theory [78] for optical waveguides where we incorporate gain and loss in the temporal evolution equations as we elucidate in the following.

It is convenient to study \textit{total} electromagnetic fields inside the cavity by resorting to electric and magnetic fields’ amplitudes, namely by two-dimensional vectors \( \mathbf{E}(z,t) = [E_1(z,t)\ E_2(z,t)]^T \) and \( \mathbf{H}(z,t) = [H_1(z,t)\ H_2(z,t)]^T \), respectively, where the vector components represent amplitudes of the total fields in WG1 and WG2. The two-dimensional vectors \( \mathbf{E} \) and \( \mathbf{H} \) are related to the wave amplitudes \( \mathbf{a}^+ \) and \( \mathbf{a}^- \) as shown in (A2) in Appendix A, by resorting to the concept of characteristic impedance of the waveguides as discussed in Appendix A (a procedure described in coupled transmission line theory [68]). The reason for adopting this \( \mathbf{E} \) and \( \mathbf{H} \) field representation is that it is straightforward to include losses and gain in the total.
Field formulation. Such coupled-wave formulation can be then readily characterized using conventional FDTD implemented by a standard Yee [79] algorithm that has been extensively studied for transmission lines [80] as well as for formulations based on both $E$ and $H$ fields [80–82]. As done in Ref. [22,47], for example, the state vector that describes the total field amplitudes is denoted by $\Psi(z,t) = [E^T(z,t) \ H^T(z,t)]^T$ and it is related to normalized wave amplitude state vector $\hat{\Psi}(z,t)$ through a matrix transformation (see Appendix A). The space-time evolution of the total field amplitude state vector $\Psi(z,t)$ in the periodic waveguide is constructed in a similar fashion to (3) and is provided in Appendix A, (A6).

We proceed by generalizing the above analysis to periodic structures made of sections of uniform coupled waveguides in the presence of gain and loss. The nonlinear gain is provided by an externally pumped active medium and is incorporated into the analysis through the polarization density amplitudes $P_1(z,t)$ and $P_2(z,t)$. These polarization densities represent the effective polarization field amplitudes induced in WG1 and WG2 because of the gain medium and depend on how much the field distributions associated to the amplitudes $E_1(z,t)$ and $E_2(z,t)$ overlap with the gain medium. Hence, generalizing (A6) to the case of gain and loss one obtains

$$\frac{\partial \Psi(z,t)}{\partial z} = -\mathbf{M}(z) \frac{\partial \Psi(z,t)}{\partial t} + \left(\frac{\sigma_e}{s}\right) \frac{\partial \mathbf{P}(z,t)}{\partial t}$$

$$-\left(\begin{array}{c} 0 \\ \mathbf{P} \end{array}\right) \frac{\partial}{\partial z} \Psi(z,t),$$

where $\mathbf{M}(z)$ is the system matrix of the lossless and gainless coupled waveguide given in (A7). The two-dimensional polarization density amplitude vector $\mathbf{P}(z,t) = [P_1(z,t) \ P_2(z,t)]^T$ represents the polarization density in WG1 and WG2 induced by the transition of the active atoms. Therefore $\mathbf{P}(z,t)$ accounts for the active material (gain), and $s$ is a $2 \times 2$ gain coupling matrix that represents the interaction strength of the gain media and the coupled waveguide fields. The $2 \times 2$ matrix $\mathbf{P}$ is a per-unit-length dielectric loss parameter, also given in Appendix A. The atomic transitions occur within a simplified, yet realistic, four-level energy atomic system shown in Fig. 4. Accordingly, the time evolution for the polarization density amplitudes in WG1 and WG2 in the presence of a forcing electric field is described by the homogeneously broadened Lorentzian oscillator model and obtained as [82–85]

$$\left[\frac{\partial^2}{\partial t^2} + \Delta \omega_e \frac{\partial}{\partial t} + \omega_e^2\right] \mathbf{P}(z,t) = -\sigma_e \Delta N \ s \ \mathbf{E}(z,t),$$

where

$$\sigma_e = 6\pi \epsilon_0 c^3 / (\tau_{21} \omega_e^2),$$

and $\tau_{21}$ is the photon lifetime of the transition between the second and the first energy states (i.e., between energy levels 2 and 1), $-e$ and $m$ are the charge and the mass of an electron, $\epsilon_0$ is the free-space permittivity, $\omega_e$ is the angular frequency of emission (chosen for our specific implementation at $\lambda_e = 1550$ nm), and $\Delta \omega_e$ is the full width at half maximum linewidth of the atomic dipolar transition. The $2 \times 2$ matrix $s$ in (5) is given in Appendix C and indicates which of the two waveguides is involved in the coupling between gain media and electric-field amplitudes. Here, its components are simply taken as 1 or 0, though it could take into account also other confinement factors. The atomic system has four energy levels with atomic population densities of $N_0$, $N_1$, $N_2$, and $N_3$ while $\Delta N$ in (5) is also the population density difference between the first and the second energy levels (i.e., $\Delta N = N_2 - N_1$) [83,84]. The time- and space-dependent population density $N_j(z,t)$ at each energy level $j = 0, 1, 2, 3$ is obtained from the nonlinear rate equations provided in Appendix A. Here, we assume that the total active material density (i.e., $N_T = \sum_{j=0}^3 N_j(z,t)$) is uniformly distributed and is invariant with space and time. The cavity is optically pumped with pumping rate $R_p$ transferring active atoms from the ground state (zeroth level) into the highest energy level (third level). The lasing action occurs when the atomic transition from the second level to the first one is slow and radiative, leading to a population inversion (i.e., lasing condition is when $\tau_{21} > \tau_{32}, \tau_{10}$ which results in $N_2 > N_1$). Here, we assume that the active material is erbium ($\text{Er}^{3+}$) and it is doped into the substrate. The pumping rate $R_p$ is a tunable parameter and can be varied by the external pump intensity in a real experiment. Therefore, we fully characterize the laser cavity by simultaneously solving the coupled set of nonlinear rate equations given in (A9) along with the wave equations in (4) and the Lorentzian equation in (5). Here, we utilize the FDTD algorithm to solve this system of equations along with proper boundary conditions. Details on the FDTD algorithm employed here are included in Appendix B. The boundary conditions for the system imply that the periodic waveguide is terminated at both sides by output waveguides (see Fig. 3) whose loading is represented by their characteristic impedances given in Appendix C (Table III). To study the operational scheme of the DBE lasers we first investigate the characteristics of the cold DBE cavity (also discussed in [47,76]), where for "cold" we mean absence of gain, however we pay attention to losses and include realistic parameters of the coupled waveguide.

**C. Steady-state gain medium response**

In order to provide insight into characteristics of the DBE laser, the response of the cold DBE cavity is investigated as
as well as the steady-state response of the DBE cavity with gain medium. Assuming time harmonic fields as $e^{-i\omega t}$, the electric field and polarization density vectors in (4) are given by $\mathbf{E}(z,t) = \mathbf{Re}[\mathbf{E}(z)e^{-i\omega t}]$ and $\mathbf{P}(z,t) = \mathbf{Re}[\mathbf{P}(z)e^{-i\omega t}]$, respectively, in which $\mathbf{E}(z)$ and $\mathbf{P}(z)$ are phasors.

In addition, we consider the active material population density at each energy level from the steady-state point of view (at the steady state we have $dN_j/dt = 0$, $j = 0,1,2,3$). As such the population difference $\Delta N$ is constant and expressed as a function of the gain medium parameters, as typically done in steady-state linearized gain models [84]. Therefore, the polarization density vector is simply related to the electric-field vector through

$$\mathbf{P}(z) = \frac{\sigma_e}{\omega^2 - \omega_e^2 + i\omega\Delta\omega_e} \Delta N \mathbf{s} \mathbf{E}(z) = \frac{g}{i\omega} \mathbf{s} \mathbf{E}(z). \quad (7)$$

This equation defines the linearized gain parameter $g = \frac{i\omega\Delta N \sigma_e}{(\omega^2 - \omega_e^2 + i\omega\Delta\omega_e)}$ with unit of S/m. This is analogous to the description where polarization density amplitudes are related to the electric-field amplitudes through the susceptibility (see definition in pages 494–541 in [83] and pages 103–108 in [84]). The parameter $g$ represents gain/loss when its real part is negative/positive, whereas its imaginary part represents reactive loading due to the gain medium, under the small signal (linear) regime. In the example investigated here, only WG2 has active material (i.e., erbium, Er$^{3+}$) and therefore only the 2,2 entry, shown in Appendix C, of the gain coupling matrix $\mathbf{s}$ is nonvanishing. Active medium parameters are detailed in Appendix C, and the plot of the gain parameter $g$, with units of S/m, is shown in Fig. 5(a) for a pumping rate of $R_p = 6 \times 10^6$ s$^{-1}$. Figure 5(a) shows that the gain parameter profile follows a Lorentzian shape with a negative real part within the frequency band of interest where, for simplicity, the periodic coupled waveguide in Fig. 3 is devised to have the DBE angular frequency $\omega_d$ coinciding with the emission angular frequency $\omega_e$.

D. Cold DBE cavity characteristics

The properties of the cold DBE cavity, i.e., without the gain medium, are described here. We investigate the transfer function and the loaded $Q$ factor of the DBE cavity, using the transfer-matrix analysis (refer to [47,65] for a thorough analysis of waveguides with DBE using the transfer-matrix method). The analysis of cold DBE cavities has been done in various references, to mention a few [66,64,67,54,47], however here we only demonstrate the principal characteristics relative to lasing and the main contributing factors to lowering the lasing threshold and the single mode property in a DBE laser.

Note that the evolution equations of the wave amplitudes in the coupled waveguides can be described with first-order differential coupled-wave equations that may be written in a Hermitian form (in the absence of gain and loss), as conventionally done in coupled-mode theory [86]. Therefore, this lossless system can be locally referred to as Hermitian (in the context of coupled-wave propagation [86,87]). In other words, the time harmonic coupled wave equations (1) are given in terms of the phasors $\mathbf{a}^\pm(z)$ with $\mathbf{a}^\pm(z,t) = \mathbf{Re}[\mathbf{a}^\pm(z)e^{-i\omega t}]$ in a frequency domain description as such,

$$\frac{\partial \mathbf{a}^+(z)}{\partial z} = ik_0 \mathbf{n} \mathbf{a}^+(z); \quad \frac{\partial \mathbf{a}^-(z,t)}{\partial z} = -ik_0 \mathbf{n} \mathbf{a}^-(z), \quad (8)$$

where $k_0 = \omega/c$. By inspecting (8), one concludes that the matrix $\mathbf{k}_0 \mathbf{n}$ is Hermitian since it is symmetric and therefore diagonalizable (refer to [44] for details) in each uniform segment of the lossless coupled waveguides.

The fundamental consequence of spatial periodicity of the lossless coupled waveguides under consideration (as those in Fig. 1) is that a nonuniform $\mathbf{n}(z)$ allows the EPD to occur even though the individual lossless uniform waveguides constitute Hermitian matrices as explained in detail in [88] using a transfer-matrix formalism. In the following results are obtained using the TL formalism in Sec. II B implemented via a transfer-matrix formalism following [66,88,89], skipping the details here.

The finite length resonator is terminated by waveguides as shown in Fig. 3 and whose characteristic impedances are given in Appendix C (Table III). We first plot in Fig. 5(b) the transfer function [66].

Sharp transmission peaks near the DBE angular frequency $\omega_d$ are observed and the peak closest to the DBE frequency has the highest $Q$ factor [47]; we refer to it as the DBE
resonance of the structure with finite length and we denote it by $\omega_{r,d}$. Note that such a peak is the sharpest one, and in Fig. 5(b) occurs at $f_{r,d} = \omega_{r,d}/(2\pi) \approx 193$ THz, i.e., at $\lambda_{r,d} = 2\pi c/\omega_{r,d} \approx 1553$ nm, and that several peaks are within the emission spectrum [Fig. 5(a)] of the gain material. The resonance frequency closest to the DBE frequency is dominant over all other resonances in the FPC with DBE. Because of the fourth power in the dispersion relation $(\omega_d - \omega)^4 \approx h(k - k_d)^4$, a FPC resonance will occur at an angular frequency $\omega_{r,d}$ extremely close to $\omega_d$, where the group velocity vanishes, hence causing a very high density of states at $\omega_{r,d}$ [42]. This leads to the FPC resonance with highest group delay and highest local density of states (LDOS) as was shown in Ref. [47]. It is because of the largest density of states associated to the resonance peak at $\omega_{r,d}$ that a single frequency operation of a laser cavity with DBE is expected, as will be shown in Sec. IV.

For the sake of completeness, we compare the transfer function result obtained using the transfer-matrix analysis with that calculated using the FDTD method that will be used later on in the paper. Figure 5(b) shows identical agreement that calculated using the FDTD method that will be used later on in the paper. Figure 6 shows the peaks of the transfer function varying the number of unit cells in the periodic structure, calculated using the transfer-matrix method. This figure shows that the transmission resonance frequencies are sharper when closer to the DBE frequency, and that the transmission resonance peak (i.e., the DBE resonance at $\omega_{r,d}$) shifts toward the DBE frequency $\omega_d$ as the number of cells $N$ increases, following the trajectories

$$\omega_{r,d} = \omega_d - \xi/N^4,$$

where $\xi = h(\pi/d)^4 \approx 64 \omega_d$. This formula serves to estimate the working DBE resonance frequency as a function of number of unit cells. It should be observed that such DBE cavities have more or less twice the number of resonance frequencies as a uniform cavity with the same length, and that they are even closer to each other than those in a uniform cavity.

Nevertheless, a single frequency of lasing operation will be demonstrated, due to the very high density of states of the resonance at $\omega_{r,d}$ [47].

In Fig. 7 we show some properties of the loaded $Q$ factor for the DBE cavity. The loaded $Q$ factor is defined as $Q_{\text{tot}} = \omega_{r,d}(W_e + W_m)/P_L$ where $W_e$ and $W_m$ are the total electric and the magnetic time-average stored energies, while $P_L$ is the total time-average power loss. In the calculation of the power loss we consider the power dissipated in the material and that received by the loads. We plot the calculated loaded $Q$ factor at the DBE resonance versus the number of unit cells in Fig. 7 where we show how it scales with $N$. (Resonance is recalculated at each length, for each case.) In particular, we observe the unusual trend of the loaded $Q$ factor where the loaded $Q$ factor increases as $N^5$ for large $N$ for the lossless DBE cavity (as shown in Fig. 7 by the fitting formula loaded $Q = a + b N^5$). However, dissipative losses in the dielectric [represented by the parameter $\gamma$ in (4) and in (A8)] limits such an anomalous trend. In other words, the loaded $Q$ factor ceases to increase at one critical cavity length at which it starts to deteriorate when losses overwhelm the response, i.e., eliminate transmission peak of the cavity. Note that for $\gamma > 17$ S/m, which is a high loss condition, the FPC with DBE composed of 20 unit cells would have negligible transmission and therefore the loaded $Q$ factor would be less than 1000. Such high loss cases are not considered here though they may have advantages in certain classes of lasers with PT symmetry, for instance [31,44]. Here, it is important to pay attention to the design of the DBE cavity by choosing the optimum number of cells to control the effect of losses and allow for the DBE resonance condition. We stress that the values of $Q$ factor (in Fig. 7) are only representative for the DBE cavity investigated here, and higher values of $Q$ could be obtained for an optimized
design considering other implementations, utilizing CROWs for instance as in [50].

III. DEGENERATE BAND EDGE (DBE) LASER

The evolution of the lasing dynamics in active DBE cavities is described by the nonlinear time domain equations stated in Sec. II and Appendix A. In particular, the transient response of lasing action is well described by the evolution of gain in the cavity. In the small signal regime, photons within the DBE resonance at \( \omega = \omega_{r,d} \) have the longest lifetime and the highest effective gain coefficient as defined in [42], among all other resonances in the cavity. As long as the overall gain experienced by such photons (electromagnetic waves) inside the cavity in a round trip is higher than the cavity losses (due to dissipative mechanism and escaping energy from the cavity ends), the intensity of the electromagnetic waves grows; and the DBE resonance has the highest growth rate among all other resonances in the cavity. Further analysis for the linear gain enhancement in DBE structures is established in [47].

On the other hand, for large electromagnetic wave intensities inside the cavity, nonlinearities are manifested in the rate equations, saturation occurs, and the output field amplitude reaches a steady state. Here, we analyze the lasing action in the optical-waveguide-based DBE laser using the FDTD algorithm (see details in Appendix B). The parameters of the DBE laser are provided in Appendix C. Note that transient and steady-state results are obtained here assuming that an initial short Gaussian pulse (whose parameters are provided in Appendix B) is launched into the WG2 from the left (Fig. 3).

The transient responses of the electric-field amplitudes at all four waveguide outputs, for the case of a lossless DBE cavity, are plotted for a pumping rate of \( R_p = 10^7 \text{ s}^{-1} \) in Fig. 8. Note that the pumping rate of \( 10^7 \text{ s}^{-1} \) used in Fig. 8 is larger than the lasing threshold pumping rate as we show later. We observe from Fig. 8 that the output field amplitudes saturate to the steady state at around \( t_s \approx 350 \text{ ps} \) thanks to the nonlinearity in the gain medium described in the nonlinear rate equations (see Appendix A). Note that the output electric-field amplitudes in the waveguide outputs oscillate at a single frequency of \( \sim 192.5 \text{ THz} \) (\( \sim \lambda = 1556.9 \text{ nm} \)) as seen in Fig. 8 bottom right panel, which is very close to the DBE cavity resonance frequency \( f_{r,d} = 193 \text{ THz} \) (\( \lambda_{r,d} = 1553 \text{ nm} \)).

We first illustrate the mode selectivity by showing the space-time field evolution inside the cavity in Fig. 9 for which the DBE resonant field concentrated at the cavity center starts to grow exponentially after \( \sim 70 \text{ ps} \). Therefore, although many FPC resonances experience gain as seen in Fig. 5, the DBE resonance experiences the highest gain and dominates the output spectrum thanks to its unique field distribution and highest \( Q \) factor. In addition, we plot the steady-state time-averaged electric-field intensity inside the DBE laser cavity in Fig. 10. The steady-state field intensity is mainly concentrated near the cavity center for WG2, while the maximum field intensity in WG1 is at least two orders of magnitude lower than that in WG2. Nevertheless, the presence of WG1 is crucial to achieve the DBE. In addition, the steady-state field profile inside the DBE laser cavity resembles the field of the DBE resonance in the cold DBE cavity (not shown here for brevity; see Refs. [47,66]). Such observation indicates that the DBE features, which pertain to lossless and gainless periodic waveguides, are still persistent even when the cavity contains a nonlinear gain. However, when operating just above the lasing threshold shown next, the main features of the DBE are retained. Very high levels of gain (i.e., very high pumping rates) may lead to other regimes of operation not considered in this paper since high gain may adversely deteriorate the DBE condition (see discussion in [47], and in [25] for RBE structures).

![transient response of the electric-field amplitudes recorded at all four waveguide outputs (Fig. 3) of the DBE laser (with \( N = 20 \) unit cells) at \( z = 0 \) and \( z = L \) for a pumping rate of \( R_p = 10^7 \text{ s}^{-1} \). The zoomed area in bottom-right panel indicates the single frequency laser operation at the steady state. The laser steady-state outputs are at a single frequency of \( \sim 192.5 \text{ THz} \) (\( \sim \lambda = 1556.9 \text{ nm} \)) which is very close to the DBE resonance frequency \( f_{r,d} = 193 \text{ THz} \) (\( \lambda_{r,d} = 1553 \text{ nm} \)).](image1)

![steady-state field intensity is mainly concentrated near the cavity center for WG2, while the maximum field intensity in WG1 is at least two orders of magnitude lower than that in WG2. Nevertheless, the presence of WG1 is crucial to achieve the DBE. In addition, the steady-state field profile inside the DBE laser cavity resembles the field of the DBE resonance in the cold DBE cavity (not shown here for brevity; see Refs. [47,66]). Such observation indicates that the DBE features, which pertain to lossless and gainless periodic waveguides, are still persistent even when the cavity contains a nonlinear gain. However, when operating just above the lasing threshold shown next, the main features of the DBE are retained. Very high levels of gain (i.e., very high pumping rates) may lead to other regimes of operation not considered in this paper since high gain may adversely deteriorate the DBE condition (see discussion in [47], and in [25] for RBE structures).]](image2)
To calculate the lasing threshold for the DBE laser using time domain simulations, we calculate the root-mean-square (rms) value of the steady-state electric-field amplitude at the end of the WG1 of the cavity as a function of the pumping rate, shown in Fig. 11. Two DBE laser cases with the 20-unit cells are investigated in Fig. 11, a lossless DBE laser with the total loaded $Q$ factor of 4200 and a lossy DBE laser with $\gamma = 13$ S/m (see Fig. 7) and a total loaded $Q$ factor of 3400. Remarkably, the DBE laser is shown to exhibit a single frequency output even when pumped up to more than 20 times of its threshold as seen in Fig. 11. The threshold pumping rate $R_p^\text{th}$ is defined as the minimum pump rate that causes instability, i.e., the electric-field amplitude to grow exponentially inside the cavity. Numerically, it is calculated as the value of $R_p$ at which the output field amplitude experiences a sudden transition from being around zero to having a significantly larger steady-state value. This is achieved by sweeping $R_p$ and observing the output steady-state field amplitudes. As in Fig. 11, the root-mean-square (rms) of the steady-state output electric-field amplitude (here recorded after 1.5 ns) for a lossless DBE laser is negligible (~0 V/m) for $R_p \sim 1.08 \times 10^8$ s$^{-1}$ while it is significantly large (~0.86 x 10$^4$ V/m) for $R_p \sim 1.22 \times 10^6$ s$^{-1}$, indicating a threshold pumping rate of approximately $R_p^\text{th} \sim 1.15 \times 10^6$ s$^{-1}$ within a maximum error of ~6% due to finite numbers of simulation points. As such, the lasing pump thresholds for the lossless and the lossy DBE lasers are $R_p^\text{th} \approx 1.15 \times 10^6$ s$^{-1}$ and $R_p^\text{th} \approx 1.4 \times 10^6$ s$^{-1}$ respectively. Furthermore, Fig. 11 shows that the rms value of the steady-state electric-field amplitude of the laser output increases linearly with the pump rate above threshold.

We also calculate, using the FDTD method, the threshold pumping rate for the lossless DBE laser with different numbers of unit cells following the same procedure we have used for calculating the lasing threshold for the DBE with $N = 20$ unit cells in Fig. 11. Then, we plot the threshold pumping rate for the DBE laser varying as a function of the number of unit cells $N$ in Fig. 12. The trend of the threshold pumping rate for the DBE laser for large $N$ (i.e., $N \geq 16$) is also shown in Fig. 12 by the asymptotic fitting curve $R_p^\text{th} \propto N^{-5}$ which agrees with the theoretical calculations [47]. The reasons for such unconventional scaling of the threshold pumping rate with length is that the loaded $Q$ factor of the DBE laser scales as $N^5$ (see [47,76] for more details), as compared to that of a conventional RBE laser which scales as $N^3$ [41], or that of a homogenous cavity that scales simply as $N$, as shown in Fig. 1(a), for the three regimes of operations discussed in this paper. We point out that for a large number of unit cells the DBE laser threshold becomes substantially lower by orders of magnitude than uniform FPC lasers as shown in Fig. 1(a). Interestingly, the constants of proportionality of the trend of $R_p^\text{th}$ as a function of $N$ are the same as those used to fit the loaded $Q$ factor in Fig. 7, indicating that $R_p^\text{th}(N) \sim Q^{-1}(N)$. It may seem apparent that the pumping threshold is only dictated by the loaded $Q$ factor, yet the field structure of the cavity plays a pivotal role in lowering the threshold, as we discuss in the following section.
FIG. 13. Schematic geometries of the three FPC laser regimes compared in this paper: (a) FPC with DBE, (b) FPC with RBE, and (c) uniform FPC. Note that the FPC with DBE does not require mirrors at its ends, while they are essential in the uniform FPC, and somewhat necessary in the FPC with RBE, to ensure high $Q$ factor.

IV. COMPARISON BETWEEN THE DBE LASER AND OTHER CONVENTIONAL LASERS

To elaborate on the reasons of the superiority aspects of the DBE laser, we establish here a comparison between the proposed DBE laser with two other regimes of laser operations: (i) uniform FPC laser and (ii) RBE laser. The geometries of the three aforementioned laser cavities are shown in Fig. 13 where the DBE and RBE laser cases are corresponding to periodic structures operating near their band edge frequencies and comprising two periodically coupled waveguides, while the uniform FPC cavity is composed of two uniform (i.e., nonperiodic) coupled waveguides. Note that the choice of coupled waveguides for the RBE and the uniform cavity lasers is not necessary, however we here use two coupled waveguides to preserve the analogy to the DBE laser for comparison purposes. Additionally, we aim at showing fundamentally unique and superior properties of DBE laser compared to the two other regimes. Therefore, the $Q$ factors and threshold values considered herein for all regimes of operation are shown as representative examples. Such performance and the conclusions drawn thereafter are yet valid when implemented in other optical platforms supporting the DBE.

The dispersion relations and the magnitude of the transmission coefficients for the RBE and the DBE representative examples considered here are shown in Figs. 14(a) and 14(b), respectively. Indeed, we consider the aforementioned cold cavities (namely the FPCs with the DBE, the RBE, and the uniform case) such that they exhibit the same resonance frequency, length, and loaded $Q$ factor (that also takes into account losses as we will show later). This is achieved by adopting, for instance, the parameters for such cavities as in Appendix C. In order to achieve the same loaded $Q$ factor for the three aforementioned cavities with the same length, the reflectivity at the cavity ends is properly chosen as follows by resorting to the useful and general concept of waveguide impedance. We define a load power reflectivity at each waveguide (i.e., at each port) as the square of the magnitude of the reflection coefficient at the interface between the output waveguide and the last FPC waveguide segment due to the difference in their characteristic impedances, as in Fig. 13 [see (C2) in Appendix C for definitions of reflectivity]. Note that the output WGs’ characteristic impedances for the three kinds of cavities as reported in Table III are varied to ensure that the three cavities have the same $Q$.

It is important to stress that the uniform cavity requires extremely high mirror reflectivity at each end (the power reflection coefficient is $\sim 0.995$, as defined in Appendix C) to have the same loaded $Q$ factor as the RBE and DBE cavities. Indeed, especially for the DBE case, reflection at the ends of the cavity is not realized by physical mirrors (i.e., large impedance mismatch between waveguide segments) since waveguide reflectivities are much lower. Indeed, a large reflection for the REB and especially for the DBE cavity is caused by mismatch between the degenerate Floquet-Bloch eigenwaves and the propagation eigenwaves in the external waveguides, and not by the simple characteristic impedance mismatch in each waveguide. Similar properties are also demonstrated in DBE CROWs [50] in which impedance mismatch or mirrors are not required for generating high $Q$-factor values.

We show in Fig. 15 the field distribution inside the cold coupled waveguides based FPCs operation at the resonance frequency for the two cases: the DBE cavity, considered in Secs. II C and III, as well as the RBE cavity. The fields are plotted only for WG2 in both DBE and RBE FPCs and sampled...
FIG. 15. Normalized electric-field amplitude inside (a) the RBE cold FPC and (b) the DBE cold FPC varying as a function of the loaded $Q$ factor of the cavity. The loaded $Q$ factor decreases by incorporating dissipative losses. The field is plotted inside WG2 in both (a) and (b), which represents the maximum field enhancement for each case (when $Q = 4200$) occurring respectively at $0.9985\omega_d$ and $0.99852\omega_d$. Note that the normalized field amplitude inside the DBE cavity is much stronger than that in the RBE cavity and is mainly concentrated in the middle of the cavity.

FIG. 16. Comparison between the lasing threshold pumping rate of the proposed DBE laser with the conventional uniform FPC and the RBE lasers of equal length ($L = 4.8 \mu m$) and resonant frequency (193 THz) varying as a function of their loaded $Q$ factor. The comparative plot clearly shows that the DBE laser has a significantly lower lasing threshold as compared to the RBE and the conventional uniform FPC lasers having the same $Q$ factor.

observation can be generalized to other implementations of FPCs with DBE and RBE.

Next, we compare the three regimes of laser operation in terms of lasing threshold namely for the DBE, RBE, and uniform cavities. Figure 16 shows a comparison between the lasing threshold pump rate $R_{th}$ of the three aforementioned cavities as the loaded $Q$ factor varies. The first and foremost observation is that the lossless DBE laser (the maximum loaded $Q$ factor of $\sim 4200$ is given by termination loading) develops much lower threshold pump rate as compared to the lossless RBE laser of equal $Q$ factor and length, which in turn is also lower than that of the corresponding lossless uniform FPC laser. When losses are considered (loaded $Q$ factor less than 4200), the DBE laser has a lower threshold for all the ranges of the loaded $Q$ factor considered in Fig. 16 (i.e., from 1000 to 4200). For example, the DBE laser has 30% lower threshold than the uniform FPC laser of equal length when the loaded $Q$ factor is $\sim 2000$.

The results presented so far for the DBE laser assumed that the DBE cavity has mirrors with reflectivity at both ends, for simplicity of calculation, since for comparison we wanted to have (i) the same length, (ii) the same resonance frequency, and (iii) and the same $Q$ factor in all three FPC types. In general, the DBE resonant field confinement shown in Fig. 15(b) occurs even without mirror reflectors. Indeed, no mirrors are required for the DBE cavity to develop high $Q$ factor and strong field enhancement [47]. This is manifestly different from conventional uniform FPC laser cavities for which mirrors with high reflectivity are needed to reduce the lasing threshold [84]. We stress that in the DBE cavity, strong reflection of the DBE eigenwaves is still present, given by the degeneracy condition and almost unmatched Floquet-Bloch eigenwave impedance with the impedance of the load WG1 and WG2 waveguides, and not by the mismatch of the individual last waveguide segments to WG1 and WG2. The mirror reflectivity defined in Appendix C does not represent the Floquet-Bloch eigenwave reflection coefficient in the DBE.
FIG. 17. Comparison between the lasing threshold pumping rate for the proposed DBE laser (dashed red) and the conventional uniform FPC (dash-dotted black) varying as a function of the mirror (load) power reflectivity. The DBE lasing threshold is significantly low regardless of the mirror reflectivity (even with no reflectivity at all) while the lasing threshold for the uniform FPC laser strongly depends on its mirror reflectivity [see definitions of reflectivity in Appendix C(C2)].

As such, if we choose a mirror reflectivity of 98% for the uniform FPC laser; the corresponding uniform FPC length must substantially increase to achieve the same threshold of the DBE laser.

We finally show in Fig. 18 the excellent mode selection scheme of the DBE laser by observing the purity of its output field intensity spectrum in comparison with a long uniform FPC laser with relatively similar lasing threshold and the same $Q$ factor, both operating at 1550 nm. The uniform FPC laser in this case is considered to have 98% mirror power reflection coefficient and also ten times longer active medium length (i.e., $L$) than the DBE case in Sec. III (both are lossless cases for simplicity, and their parameters are provided in Appendix C).

The load impedance of the DBE cavity ends is assumed to be the same as given in Appendix C (Table III). The length of the uniform FPC laser is chosen such that both the DBE and the uniform FPC lasers have an equal lasing threshold of the order of $10^6 \text{s}^{-1}$, and also a similar $Q$ factor of $\sim 4000$. Note that both cavities have multiple resonances within the gain emission spectrum.

We plot in Fig. 18 the normalized output steady-state field intensity spectrum for the DBE and uniform FPC lasers, both pumped above their respective threshold with $R_p = 10^7 \text{s}^{-1}$ for both cases. The long cold uniform FPC has multiple resonances within the gain emission spectrum with a small free spectral range, i.e., spectral separation of the FPC resonance frequencies of $\sim 1.6 \text{THz}$. Moreover, such resonances have the same spectral width, therefore they experience comparable gains, as discussed in [31,84]. This will in turn lead to multiple frequencies of oscillation within the laser (this is a typical scenario in conventional lasers; see page 42 in [30]). Indeed, the steady-state output field intensity spectrum for the uniform FPC laser plotted in Fig. 18 shows multiple resonances within the gain emission spectrum.
frequencies associated with the multiple lasing modes excited cavity (the resonant frequencies of the FPC without gain are depicted by dashed vertical lines in Fig. 18). On the contrary, the output intensity spectrum for the DBE laser seen in Fig. 18(a) shows a well-defined single-frequency (i.e., single oscillating mode) operation at ∼192.5 THz near the cold DBE resonance frequency $f_{rd} \sim 193$ THz (as discussed earlier in Sec. II and Fig. 8). This indicates that the field of the DBE FPC resonance experiences a substantial gain contrast against all other FPC resonances (see Fig. 5), which leads to low-threshold lasing and mode selectivity when operating near the DBE. The remarkable single-frequency operation of the DBE low-threshold laser demonstrates its robustness as well as practicality for realizing single frequency coherent and low-noise sources.

V. CONCLUSION

We have demonstrated a regime in low-threshold lasers based on an exceptional point of degeneracy referred to as the degenerate band edge (DBE) in a pair of coupled periodic waveguides. We have provided the underlying theory behind such a regime of operation of lasers that utilizes EPDs. In particular, we have demonstrated that the DBE laser features a significantly lower lasing threshold as compared to its conventional counterparts, i.e., a RBE laser and a uniform FPC laser, having the same resonant frequency, total length, and loaded gain contrast factor. The time domain simulation results have also shown that the threshold pumping rate for the DBE laser exhibits an unprecedented scaling law of the threshold with length as $N^{-3}$, where $N$ is the number of unit cells. Nonlinear gain media inclusion in the DBE cavity does not significantly perturb such mathematical condition, and the presented results have shown that the DBE laser can be pumped up to 20 times the threshold value and still maintain a single mode and a structured field distribution similar to that in the cold DBE cavity. The DBE laser does not require mirrors at its ends; indeed in some provided examples the mirrorless DBE cavity was terminated on a pair of waveguides identical to the ones inside the cavity. Furthermore, we have demonstrated the mode selection scheme associated with the DBE that leads to a single-frequency operation of the DBE laser in contrast to the conventional lasers that may operate with multiple oscillating frequencies. The concepts proposed, and phenomenological conclusion drawn here, can be readily applied to provide robust threshold conditions, high efficiency, and high output power, as well as low phase noise of not only optical but also infrared and terahertz sources.

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APPENDIX A: COUPLED WAVES FORMULATION AND RATE EQUATIONS

The evolution of the real-valued spatiotemporal wave amplitudes $a^+(z,t)$ and $a^-(z,t)$ are provided in (1)–(3) for a lossless and gainless uniform coupled waveguide. The system evolution matrix $\hat{M}$ in (3) is obtained from (2) and it is a $4 \times 4$ matrix given by

$$\hat{M} = \begin{bmatrix} \frac{1}{c} & n & 0 & -n \\ 0 & -\frac{1}{c} & -n & n \end{bmatrix}. \quad (A1)$$

In this paper, it is convenient to resort to a fundamental description of waves propagating in the two coupled waveguides WG1 and WG2 (Figs. 1 and 3) using real-valued spatiotemporal field amplitudes $E_i(z,t)$, $H_i(z,t)$ and $E_2(z,t)$. $H_2(z,t)$, respectively. As discussed in Sec. II it is convenient to use the two-dimensional vectors $E(z,t) = [E_1(z,t), E_2(z,t)]^T$ and $H(z,t) = [H_1(z,t), H_2(z,t)]^T$. Note that wave amplitudes $a^+$ and $a^-$ are related to the total field amplitudes through a multiplication involving the characteristic impedance matrix as (see page 39 in [90] and also [74])

$$a^+ = \frac{1}{2} \sqrt{Z}^{-1} [E + ZH], \quad a^- = \frac{1}{2} \sqrt{Z}^{-1} [E - ZH], \quad (A2)$$

where $Z$ is the $2 \times 2$ characteristic impedance matrix,

$$Z = \begin{bmatrix} Z_{11} & Z_m \\ Z_m & Z_{22} \end{bmatrix}. \quad (A3)$$

and the characteristic impedance here is a symmetric and positive-definite matrix of the coupled waveguide (considered to piecewise uniform along $z$ and without gain or loss) which is defined using couple transmission line theory as in [78,91]. We recall that $Z$ is defined as $Z = \tilde{T}_E \tilde{T}_H^{-1}$ where $\tilde{T}_E$ and $\tilde{T}_H$ are $2 \times 2$ matrices identified as the similarity transformations that bring the electric- and magnetic-field amplitude vectors, respectively, to their eigenwave, or decoupled, form (see details in pages 94–100 in [91]). Therefore if we neglect dispersion (as discussed in Chap. 8 in [78] for instance) we can assume that $Z$ is a real matrix and is used as a multiplier in (A2). Indeed, in a frequency domain description, we assume that the dispersion of $Z$ and $n$ is totally negligible compared to other dispersions in the system, i.e., those introduced by periodicity and the gain medium in the narrow frequency spectrum investigated here. Hence the refractive index and the characteristic impedance are purely real, and the normalization in (A2) is done through right-multiplying the spatiotemporal field amplitudes vectors with the square root and inverse of the impedance matrix $\tilde{Z}$. Since the characteristic impedance matrix $Z$ is positive definite; the square roots taken in (A2) are defined as the unique positive-definite square root of the matrix $\tilde{Z}$. Recall that a positive definite matrix has a unique positive definite square root which is obtained via diagonalization of the matrix and then by taking the principal square root (positive real part) of its real and positive eigenvalues; see Chap. 11 in [92].

It is then convenient to use the total field amplitude state vector $\Psi(z,t)$ defined as $\Psi(z,t) = [E^T(z,t), H^T(z,t)]^T$. Transformation (A2) allows us to represent the state vector $\Psi(z,t) = Z^{-1/2} a^+$.
\([\mathbf{a}^+ (z,t)]^T (\mathbf{a}^- (z,t))^T \mathbf{1}^T\) in (3) by a simple transformation
\[
\mathbf{\Psi} = \mathbf{U} \hat{\mathbf{\Psi}}. \tag{A4}
\]

The \(4 \times 4\) matrix \(\mathbf{U}\) is a transformation of the physical electric- and magnetic-field amplitudes to the normalized wave (scattering) amplitudes which follows from (A2) and is given by
\[
\mathbf{U} = \begin{bmatrix}
\sqrt{\frac{Z}{n}} & \sqrt{\frac{Z}{n}} \\
\sqrt{\frac{n}{Z} - 1} & -\sqrt{\frac{n}{Z} - 1}
\end{bmatrix}.
\tag{A5}
\]

The corresponding system equation for the field amplitude state vector \(\mathbf{\Psi}(z,t)\) is given by
\[
\frac{\partial \mathbf{\Psi}(z,t)}{\partial z} = -\mathbf{M} \frac{\partial \mathbf{\Psi}(z,t)}{\partial t}. \tag{A6}
\]

Using the transformation (A3) between \(\mathbf{\Psi}(z,t)\) and \(\hat{\mathbf{\Psi}}(z,t)\) we thereby find the system matrix \(\mathbf{\hat{M}}\) in (A2) and (4) in terms of the other system matrix \(\mathbf{M}\) in (3) as
\[
\mathbf{M} = \mathbf{U} \mathbf{\hat{M}} \mathbf{U}^{-1} = \frac{1}{c} \begin{pmatrix} 0 & \sqrt{\frac{Z}{n} - 1} & \sqrt{\frac{Z}{n} - 1} & 0 \\
\sqrt{\frac{n}{Z} - 1} & 0 & 0 & \sqrt{\frac{n}{Z} - 1} \end{pmatrix}. \tag{A7}
\]

Further discussion about such transformations is found in [93]. In general, matrices \(\mathbf{Z}\) and \(\mathbf{n}\) do not necessarily commute. However, based on our choice of parameters for all the cases provided in this paper, matrices \(\mathbf{n}\) and \(\mathbf{Z}\) commute (i.e., \(\mathbf{n} \mathbf{Z} = \mathbf{Z} \mathbf{n}\)), which also means that \(\sqrt{\frac{Z}{n} - 1} \mathbf{Z} = \mathbf{n} \mathbf{Z}\). Indeed, the reason of why matrices \(\mathbf{n}\) and \(\mathbf{Z}\) commute is because the two coupled waveguides in each segment of the periodic cell are identical, as shown in Appendix C. In other words, the diagonal entries of matrices \(\mathbf{Z}\) and \(\mathbf{n}\) for the coupled segments of the waveguide are equal, therefore \(\mathbf{n} \mathbf{Z}\) and \(\mathbf{Z} \mathbf{n}\) commute. For the uncoupled segments, \(\mathbf{Z}\) and \(\mathbf{n}\) are diagonal matrices therefore they commute. The coupled waveguide analysis in this paper can be also represented using conventional coupled-mode theory [69–74], however this is outside the scope of this paper, and as mentioned in Sec. II B we found it convenient to use the time domain transmission line formulation in (4) that is readily implemented in the FDTD [79–82].

The load waveguides on the left and right sides of the cavity are assumed not coupled to each other and they are also modeled using their characteristic impedances, which can be cast as the diagonal elements of a diagonal matrix \(\mathbf{Z}_{\text{c}}\). The load waveguides at the right and left ends of the cavity can be different from each other.

Gain and losses in the system are included via the polarization amplitude \(\mathbf{P}\) and \(\mathbf{y}\) in (4) that describes the TD evolution, and not via the \(\mathbf{\hat{M}}\) and \(\mathbf{M}\) matrices in (3) and (4). The per unit length loss parameter \(\mathbf{y}\) in (4) is a \(2 \times 2\) matrix given by
\[
\mathbf{y} = \begin{pmatrix} \gamma & 0 \\
0 & \gamma \end{pmatrix}. \tag{A8}
\]

where \(\gamma\) has the unit of Siemens/m and represents per-unit-length loss in the coupled waveguides. In a transmission line formalism, each parameter \(\gamma\) represents the per-unit-length shunt conductance. The gain media is represented by a four-level energy system as schematically shown in Fig. 4. The dynamics of the population densities of different energy levels are dictated by the nonlinear rate equations given below. In this regard, the population density of each level denoted as \(N_j\) (with \(j = 0, 1, 2, 3\)) is space and time dependent and given by [82,84,94]

\[
\begin{align*}
\frac{\partial N_3(z,t)}{\partial t} &= N_0(z,t) \frac{R_p}{\tau_{32}} - N_3(z,t), \\
\frac{\partial N_2(z,t)}{\partial t} &= N_3(z,t) \frac{1}{\hbar \omega_e} [\mathbf{s} \mathbf{E}(z,t)]^T \frac{\partial \mathbf{P}(z,t)}{\partial t} - N_2(z,t), \\
\frac{\partial N_1(z,t)}{\partial t} &= N_2(z,t) \frac{1}{\hbar \omega_e} [\mathbf{s} \mathbf{E}(z,t)]^T \frac{\partial \mathbf{P}(z,t)}{\partial t} - N_1(z,t), \\
\frac{\partial N_0(z,t)}{\partial t} &= N_1(z,t) \frac{1}{\tau_{10}} - N_0(z,t) \frac{R_p}{\tau_{32}}, \tag{A9}
\end{align*}
\]

where \(T\) denotes matrix transpose and \([\mathbf{s} \mathbf{E}(z,t)]^T \frac{\partial \mathbf{P}(z,t)}{\partial t} = \sum_{\mu \nu} s_{\mu \nu} E_{\mu} \frac{\partial P_{\nu}}{\partial t}\) and \(\hbar\) is the reduced Planck’s constant. The sum of the population densities is also equal to the total active material doping density, \(N_T = \sum_{j=0}^{3} N_j\). The waveguide and gain medium physical parameters used in the examples in this paper are given in Appendix C.

\section*{APPENDIX B: FDTD ALGORITHM FOR COUPLED WAVEGUIDE CAVITY WITH GAIN}

In this Appendix, we describe the FDTD algorithm used in this paper for the analysis of the coupled waveguide interacting with gain media featuring the space-time evolution of the waveguide electric and magnetic fields’ amplitudes \(\mathbf{E}(z,t) = [E_1(z,t) \ E_2(z,t) \ E_3(z,t)]^T\) and \(\mathbf{H}(z,t) = [H_1(z,t) \ H_2(z,t) \ H_3(z,t)]^T\). We discretize the computational domain (time and space) based on the Yee algorithm [79,80], in one dimension, such that the electric-field amplitude is stored at integer node positions (in the \(z\) direction) while staggered by \(\Delta t / 2\) in time, namely \(E_i^{n+1/2} = E_i(i \Delta z,(n + 1/2) \Delta t)\). Here, \(i\) and \(n\) are integers and \(\Delta z\) and \(\Delta t\) are grid intervals in space and time, respectively. The magnetic-field amplitude on the other hand is stored at integer times while staggered by \(\Delta z / 2\) in the \(z\) direction, i.e., \(H_i^{n+1/2} = H_i((i + 1/2) \Delta z,(n+1) \Delta t)\). The population density and polarization density amplitude vectors are also sampled at the same locations and times as the electric-field amplitude. Using the central difference approximation, the time-dependent differential equations in (4) and (5) are, respectively, written in the discrete form as

\[
\begin{align*}
E_i^{n+1/2} &= -c \Delta t (\mathbf{n}^{-1} \mathbf{Z}) (\mathbf{H}_i^{n+1/2} - \mathbf{H}_i^{n-1/2}) / \Delta z \\
&= -c (\mathbf{n}^{-1} \mathbf{Z}) (\mathbf{P}_i^{n+1/2} - \mathbf{P}_i^{n-1/2})/ \Delta t - c \Delta t (\mathbf{n}^{-1} \mathbf{Z}) \gamma_i (E_i^{n+1/2} + E_i^{n-1/2})/2 + E_i^{n-1/2}, \\
H_i^{n+1/2} &= -c \Delta t (\mathbf{n} \mathbf{Z})^{-1} (\mathbf{E}_i^{n+1/2} - \mathbf{E}_i^{n-1/2}) / \Delta z + \mathbf{H}_i^{n}. \tag{B1}
\end{align*}
\]
and
\[
P_{i}^{n+1/2} = \frac{2 \Delta t^2}{2 + \Delta \omega \Delta t} \left[ -\sigma_e \Delta N_i^{n-1/2} / \Delta t \right. \\
+ (2 / \Delta t^2 - \omega_p^2) P_i^{n-1/2} \\
+ (\Delta \omega \Delta t - 2) / (2 \Delta t^2) P_i^{n-3/2}. \quad \text{B2}
\]

Accordingly, the nonlinear rate equations for the four-level atomic system [given in (A9)] are also discretized as

\[
N_{n+1/2}^{3,i} = B_3 \left( 2 R_3 N_{0,1}^{n-1/2} + A_3 N_{3,i}^{n-1/2} \right), \\
N_{2,i}^{n+1/2} = B_2 \left[ (N_{3,i}^{n+1/2} + N_{3,i}^{n-1/2}) / \tau_{32} + A_2 N_{2,i}^{n-1/2} \right. \\
+ (E_i^{n+1/2} + E_i^{n-1/2})^T (P_i^{n+1/2} - P_i^{n-1/2}) / (\hbar \omega \Delta t), \\
N_{1,i}^{n+1/2} = B_1 \left[ (A_{n,i}^{n+1/2} + N_{2,i}^{n-1/2}) / \tau_{21} + A_1 N_{1,i}^{n-1/2} \right. \\
- (E_i^{n+1/2} + E_i^{n-1/2})^T (P_i^{n+1/2} - P_i^{n-1/2}) / (\hbar \omega \Delta t), \\
N_{0,i}^{n+1/2} = \zeta \left[ (N_{1,i}^{n+1/2} + N_{1,i}^{n-1/2}) / \tau_{10} + N_{0,i}^{n-1/2} (2 - \Delta t R_p) / \Delta t \right], \quad \text{B3}
\]

where

\[
\zeta = \frac{\Delta t}{2 + \Delta t R_p}; \quad A_m = (2 \tau_{m,m-1} - \Delta t) / (\Delta t \tau_{m,m-1}); \\
B_m = \frac{\tau_{m,m-1} \Delta t}{2 \tau_{m,m-1} + \Delta t}; \quad m = 1, 2, 3. \quad \text{B4}
\]

Explicit equations (B1)–(B3) compose the complete FDTD update equations for a coupled waveguide with an optically pumped four-level gain medium. The computation is a three-step recursive process: (i) the polarization density amplitude vector is calculated through (B2), (ii) the electric- and magnetic-field amplitudes are calculated using (B1), and (iii) the population density at each energy level is then calculated from (B3). This three-step recursive process is repeated until the end of the simulation time.

APPENDIX C: PARAMETERS OF COUPLED WAVEGUIDES AND GAIN MEDIA

The periodic waveguide has been developed to have a DBE at frequency \( \omega_d = \omega_d / (2\pi) = 193.5 \text{ THz} \) (i.e., free-space wavelength \( \lambda_d = 1550 \text{nm} \)), which coincides with the dipolar emission frequency of the active material in WG2 \( \omega_d = 193.5 \text{ THz} \). The other periodic coupled waveguide used for comparison throughout the paper is developed to have an RBE at the same frequency. Throughout the paper, we have assumed that the gain media is present only in WG2 and only the field in WG2 interacts with the gain medium. Therefore, the matrix \( \mathbf{g} \) in (4) and (7) is taken as

\[
\mathbf{g} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad \text{C1}
\]

The unit cell in the periodic structure in Figs. 1, 3, and 13 is made of two segments. The characteristic impedance matrix \( \mathbf{Z} \) and refractive index matrix \( \mathbf{n} \) of the first (uncoupled) segment,

\[
\begin{array}{c|c|c|c}
\text{Uncoupled section} & \text{Coupled section} \\
\hline
n_{11} & n_{12} & n_{11} = n_{22} & n_m \\
\hline
\text{DBE} & 2.81 & 2.81 & 2.03 & 1.32 \\
\text{RBE} & 2.2 & 2.98 & 2.47 & 0.49 \\
\text{Uniform} & 2.6 & 2.6 & 1.65 & \\
\end{array}
\]

and the second (coupled) segment (see Figs. 1 and 13) of the constitutive lossless coupled waveguides with DBE and RBE are, respectively, given in Tables I and II. These parameters can be realized using silicon ridge waveguides [95–97] using the geometry in Fig. 19. The parameters for the uniform coupled waveguide used for comparison are also provided in Tables I and II.

As an illustrative example, just to prove the proposed DBE laser concepts, the lengths of the uncoupled and the coupled sections (see Fig. 1) are \( d_1 / 2 = 2d_2 = 0.096 \mu \text{m} \). The DBE and RBE laser cavities are here composed of 20 unit cells with period of \( d = d_1 + d_2 = 0.24 \mu \text{m} \) and have a total length of \( L = 4.8 \mu \text{m} \). In our examples, we have taken the parameters in Tables I–III to model silicon ridge waveguides on a SiO2 substrate [as depicted schematically in Fig. 1(a) as well as in Ref. [78]]. The cross section of each uncoupled waveguide is shown in Fig. 19(a) and is composed of a silicon rectangular ridge of width \( w \) and height \( h \), while the coupled waveguides’ cross section is shown in Fig. 19(b) and is composed of two identical silicon rectangular ridges in proximity, with lateral gap spacing of \( g \). For example, in the DBE coupled waveguide geometry, the two uncoupled waveguides are identical with \( w = 520 \text{ nm} \) and \( h = 400 \text{ nm} \); moreover, the coupled waveguides are also identical and their parameters are \( w = 800 \text{ nm} \), \( h = 400 \text{ nm} \), and \( g = 250 \text{ nm} \). The corresponding parameters in Tables I–III are then obtained based on full-wave simulations of the coupled silicon ridge waveguides, from which the effective refractive indices as well as the characteristic impedances are extracted (full-wave simulations were carried out using CST Microwave Studio, a frequency domain solver based on the finite-element method).

The left and right mirror power reflectivities associated to the loading waveguides at the two ends of the cavity (Fig. 13) are calculated as follows. We consider the impedances of the loads of WG1 and WG2 (i.e., \( Z_{L,1}^{e} \) and \( Z_{L,2}^{e} \) at the right end

\[
\begin{array}{c|c|c|c|c}
\text{Uncoupled section} & \text{Coupled section} \\
\hline
Z_{11}^{e} (\Omega) & Z_{22}^{e} (\Omega) & Z_{11}^{e} = Z_{22}^{e} (\Omega) & Z_{c}^{e} (\Omega) \\
\hline
\text{DBE} & 159 & 108 & 272 & 187 \\
\text{RBE} & 159 & 108 & 155 & 44 \\
\text{Uniform} & 272 & 187 & \\
\end{array}
\]
Therefore, the power reflectivities can be readily calculated for WG2 (in other words right and left load impedances are equal for WG1 as well as uncoupled waveguides and load impedances for the DBE, the parameters in Tables I–III are based.

of the cavity, and \( Z'_{e1} \) and \( Z'_{e2} \) at the left end of the cavity) and the characteristic impedance entries of the first and last segments of the two waveguides upon which the cavity is terminated (i.e., \( Z'_{c1} \) and \( Z'_{c2} \) for the coupled segment and \( Z'_{u1} \) and \( Z'_{u2} \) for the uncoupled one). The superscripts \( c \) and \( u \) denote the coupled and uncoupled segments respectively, while \( r \) and \( l \) denote right and left ends of the cavity, respectively. Accordingly, we define the four mirror power reflectivity terms as

\[
R'_r = \left| \frac{Z_{l1} - Z'_{u1}}{Z_{l1} + Z'_{u1}} \right|^2, \quad R'_l = \left| \frac{Z_{l1} - Z'_{u1}}{Z_{l1} + Z'_{u1}} \right|^2,
\]

\[
R''_r = \left| \frac{Z_{l2} - Z'_{u2}}{Z_{l2} + Z'_{u2}} \right|^2, \quad R''_l = \left| \frac{Z_{l2} - Z'_{u2}}{Z_{l2} + Z'_{u2}} \right|^2,
\]

where \( R'_r \) and \( R'_l \) are the power reflectivities of the WG1 and WG2 respectively, at the right end of the cavity, whereas \( R''_r \) and \( R''_l \) are the power reflectivities of the WG1 and WG2 respectively, at the left end of the cavity.

In all the cases reported in this paper (except for those in Fig. 17), the characteristic impedances of the coupled and uncoupled waveguides and load impedances for the DBE, RBE, and uniform FPCs (see Fig. 13) are reported in Tables I–III. Therefore, the power reflectivities can be readily calculated from (C2) for each laser cavity. Moreover, the load waveguides of WG1 and WG2 are different from each other, while the right and left load impedances are equal for WG1 as well as for WG2 (in other words \( Z'_{e1} \neq Z'_{e2} \) while \( Z'_{l1} = Z'_{l2} \) and \( Z'_{u2} = Z'_{u2} \)) and such differences are used to tailor the \( Q \) factors and the thresholds of the three types of laser cavities and for comparison purposes. Note that the uniform FPC is symmetric from both right and left ends, therefore we have \( R'_r = R'_l \) and \( R''_r = R''_l \).

![Cross section of (a) uncoupled (single), and (b) coupled waveguides with two identical rectangular Silicon ridges](Image)

**FIG. 19.** Cross section of (a) uncoupled (single), and (b) coupled waveguides with two identical rectangular Silicon ridges on which the parameters in Tables I–III are based.

**TABLE III.** Characteristic impedances of the output WG1 and WG2 coupled to the laser cavities with DBE, RBE, and uniform one; both right and left ends of the cavity are assumed to be have equal load impedances in all cases of the paper, except for the results in Fig. 17.

<table>
<thead>
<tr>
<th>Load waveguides impedance</th>
<th>( Z'<em>{l1} = Z'</em>{u1} ) (kΩ)</th>
<th>( Z'<em>{l2} = Z'</em>{u2} ) (kΩ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DBE</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>RBE</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>Uniform</td>
<td>375</td>
<td>260</td>
</tr>
</tbody>
</table>

Only for the results presented in Fig. 17 is the mirror reflectivity varied in the DBE cavity, as well as for the uniform laser cavity, contrary to those in Table III. To produce the results in Fig. 17, load impedances for WG1 and WG2, for both left and right ends of the cavities, are chosen such that the reflectivity of all four mirrors is the same. In other words, for Fig. 17 we chose \( R'_r = R'_l = R''_r = R''_l \) for both cases of the DBE and the uniform FPC laser cavities which is then varied and the corresponding lasing threshold is plotted for both cases in Fig. 17. Note that the mirror reflectivity values considered in this paper can be readily realized using silicon ridge waveguides [95,96].

The gain is provided by having optically pumped active atoms (e.g., here Er\(^{3+}\) as described in [70]) doped in the WG2 cavity segment of the coupled waveguide. The photon lifetime of the transitions between the energy levels in the Er\(^{3+}\) (see Fig. 4) are \( \tau_0 = 0.1 \) ps, \( \tau_1 = 300 \) ps and \( \tau_2 = 0.1 \) ps [51]. The emission frequency and gain bandwidth are also \( f_c = 193.5 \) THz and \( \Delta f_c = 5 \) THz, respectively, and the initial ground-state electron density in the WG2 material is \( N_0 = 5 \times 10^{23} \) m\(^{-3}\) [90]. These gain medium parameters are assumed to be constant and independent of the lasing process.

In our FDTD simulations, we chose the space discretization step of \( \Delta z = 5.99 \) nm and the time discretization step is set to be \( \Delta t = 3.176 \times 10^{-6} \) ps, which is sufficiently small to have a numerically stable FDTD algorithm. The oscillation process is also initiated by launching a short Gaussian pulse of peak amplitude 1 V/m, full width at half maximum (FWHM) of \( \Delta t \) × 10\(^7\) (where \( \Delta t \) is the FDTD grid interval in time), peak time of 1.3 × FWHM, and modulated at the DBE wavelength (\( \lambda = 1550 \) nm), into the cavity.

---
